

**The Cambridge Controversies in the Theory of Capital:  
Revisiting the Reswitching Puzzle**

Michael J. Osborne \*

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**Abstract**

A solution is proposed to the reswitching puzzle.

When two techniques of production are compared reswitching can occur. Reswitching is when a technique begins by being cheapest at a low interest rate, switches to being more expensive at a higher rate, and then reswitches to being cheapest at yet higher rates. Some believe the inconsistency undermines the foundations of neoclassical economics.

The time value of money (TVM) equation is at the core of the reswitching puzzle. The equation takes the form of an  $n^{\text{th}}$  order polynomial having  $n$  roots (interest rates). In most economic and financial analyses only one root is used. The remaining  $(n-1)$  roots, mostly complex or negative, are usually ignored.

The approach in this article employs all  $n$  solutions for the interest rate in a new expression relating differences in the dependent variable to differences in interest rates. The analysis is applied to the Sraffa-Pasinetti example of reswitching. The new expression provides a different perspective on the TVM equation. From this perspective, reswitching does not occur.

The ‘multiple-interest-rate’ approach provides insights into issues other than reswitching. The issues include the NPV versus IRR debate in capital budgeting, and the quest for an accurate equation for duration in bond mathematics. Reswitching is only one problem in a class of similar problems having the TVM equation at their core.

\* email: [m.osborne@shu.ac.uk](mailto:m.osborne@shu.ac.uk)

Sheffield Business School, Sheffield Hallam University,  
Howard Street, Sheffield, S1 1WB, UK

## **The Cambridge Controversies in the Theory of Capital: Revisiting the Reswitching Puzzle**

### **1 Introduction**

In this article, a solution is proposed to a puzzle in economic theory: reswitching. The solution is provided by ‘multiple-interest-rate’ analysis. The analysis has implications for other areas of economics and finance.

The reswitching puzzle is part of the Cambridge controversies in capital theory. The controversies surfaced at the beginning of the twentieth century, intensified into the ‘Cambridge Controversies’ during the 1960s, and have simmered since. A high point of the debate is the symposium on reswitching in the *Quarterly Journal of Economics* (QJE) in 1966<sup>1</sup> containing six articles on the topic by Bruno *et al.*, Garegnani, Levhari and Samuelson, Morishima, Pasinetti, and Samuelson. A comprehensive survey of the controversies is in Harcourt (1972). Cohen and Harcourt (2003) is a recent review.

When two techniques of production are compared, reswitching is the possibility that one technique can be cheapest at a low interest rate, switch to being more expensive at a higher rate, and reswitch to being cheapest at even higher rates. For some, this inconsistency undermines the foundations of neoclassical economic theory.

Samuelson (1966) expressed his concern about reswitching thus:

*‘The phenomenon of switching back at a very low interest rate to a set of techniques that had seemed viable only at a very high interest rate involves more than esoteric technicalities. It shows the simple tale told by Jevons, Bohm-Bawerk, Wicksell, and other neoclassical writers ... cannot be universally valid.’*

Nearly forty years later, Cohen and Harcourt (2003) agree that reswitching causes problems for neoclassical economics.

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<sup>1</sup> Paradoxes in Capital Theory: A Symposium, *Quarterly Journal of Economics*, 80(4), Nov. 1966

*'Looking back over this intellectual history, Solow (1963, p.10) suggested that "when a theoretical question remains debatable after 80 years there is a presumption that the question is badly posed – or very deep indeed." Solow defended the "badly posed" answer, but we believe that the questions at issue in the recurring capital controversies are 'very deep indeed.'*" Cohen and Harcourt (2003)

This article contains a new approach to the puzzle. The analysis of reswitching employs the time value of money (TVM) equation. The TVM equation is a key equation in economics and finance. It takes the form of an  $n^{\text{th}}$  order polynomial having  $n$  roots (interest rates). In most economic and financial analyses, including the reswitching debate, it is usual to employ only one root, namely the root yielding a positive, real interest rate. The remaining  $(n-1)$  roots are mostly complex or negative, and are usually ignored. When not ignored, the unorthodox roots are seen as a problem. One of the earliest examples of the latter is Lorie & Savage (1955); one of the most recent is Brealey *et al.* (2009).

In this article it is argued that all roots (interest rates), including the unorthodox, can be employed to shed light on the reswitching puzzle. Far from being a problem, multiple interest rates are part of the solution.<sup>2</sup>

It is stressed at the outset that the analysis is within the framework of comparative statics, as it was in the 1966 Symposium.

*'Following Joan Robinson's strictures that it is most important not to apply theorems obtained from the analysis of differences to situations of change ..., modern writers usually have been most careful to stress that their analysis is essentially the comparisons of different equilibrium situations one with another and that they are not analyzing actual processes.'*  
Harcourt (1972, p.122).

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<sup>2</sup> Harcourt (1972) notes the contribution of Bruno *et al.* (1966). They mention multiple roots in the context of the reswitching debate, as do Hagemann and Kurz (1976). However, their contributions refer only to the possibility of multiple *real* roots; they do not analyze all roots, including *complex* roots.

The behavior described by Harcourt applies here. Incorporation of the passage of time into the analysis remains a challenge. However, the suggested reinterpretation of the comparative statics may provide a new route to the dynamics.

Section 2 describes a numerical example of reswitching taken from one of the contributions to the QJE symposium: Pasinetti (1966). Section 3 recalls a well-known result about the factorization of polynomials. In section 4 the result is applied to the Sraffa-Pasinetti example. A new expression is derived for differences in wage ratios resulting from differences in rates of interest (profit).<sup>3</sup> The expression employs explicitly all possible interest rates, not just the orthodox. The new perspective provided by the expression shows reswitching is no longer a concern. Section 5 contains some general discussion about the analysis. The final section is the conclusion.

## **2 An example of reswitching: Sraffa-Pasinetti**

The numerical example in Pasinetti (1966) is adapted from Sraffa (1960).<sup>4</sup> Pasinetti refutes the attempt by Levhari (1965) to demonstrate reswitching cannot happen. Levhari and Samuelson (1966) and Samuelson (1966) accept Pasinetti's refutation and admit the possibility of reswitching. Pasinetti's refutation, however, is not the end of the story.

The full details of the Sraffa-Pasinetti model are not presented here; instead the focus is on the particular analysis that Pasinetti uses to demonstrate the existence of reswitching. He creates two economic systems, *a* and *b*, each of which possesses a relationship between the wage rate and the rate of interest:

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<sup>3</sup> Both words 'profit' and 'interest' are used in this context in the reswitching literature. To avoid confusion and repetition, from this point onwards the word 'interest' is employed.

<sup>4</sup> Velupillai (1975) reports a 'general consensus that the phenomenon of reswitching of techniques was first brought to the attention of Academic Economists by Joan Robinson, David Champernowne, and Piero Sraffa.' But Velupillai notes that Fisher (1907, pp. 352-353) contains an example of reswitching. The example is brief and contained in an appendix but it is unmistakably reswitching and, moreover, part of a critique of Bohm-Bawerk's methodology. Velupillai acknowledges that Fisher did not draw out the implications of the phenomenon as Sraffa and his colleagues did; therefore the consensus remains.

$$w_a = \frac{1 - 0.8(1+r)}{20(1+r)^8} \quad (1)$$

and

$$w_b = \frac{1 - 0.8(1+r)}{(1+r)^{25} + 24} \quad (2)$$

*'Since the wage rate in 'a' and the wage rate in 'b' are expressed in terms of the same physical commodity ... the two technologies can now be compared. Clearly, on grounds of profitability, that technology will be chosen which – for any given wage rate – yields the higher rate of [interest]. Or alternatively (which comes to the same thing) that technology will be chosen which – for any given rate of [interest] – yields the higher wage rate.'*

*In order to find this out, it is sufficient to compute the values of  $w_a$  and  $w_b$ , in expressions ... (1) and (2) ..., for any given level of  $r$ .'*  
(Pasinetti, 1966)

Pasinetti's Fig.1 (not shown here but given on p. 507 of the original work) has  $w_a$  and  $w_b$  on the vertical axis and  $r$  on the horizontal axis. It shows ...

*'... the curves representing  $w_a$  and  $w_b$  intersect each other three times. There are three distinct levels of the rate of [interest], namely ~3.6 per cent, ~16.2 per cent, and 25 per cent, at which  $w_a = w_b$ , i.e., at which the two technologies are equally profitable. These three points of intersection correspond to the switching from one technology to the other as the rate of [interest] is increased from zero to its maximum.'*

In this article Pasinetti's result is displayed slightly differently. Let  $w = w_a/w_b$ . Combine Eqs. (1) and (2) to produce Eq. (3).

$$w = \frac{(1+r)^{25} + 24}{20(1+r)^8} \quad (3)$$

Fig. 1 employs Eq. (3) to display the result; it is a variant of Pasinetti's Fig.1. The ratio  $w = w_a/w_b$  is on the vertical axis (Pasinetti displays  $w_a$  and  $w_b$  separately) and  $r$  is on the horizontal axis. The range of  $r$  is 0% to 25%. The

curve crosses the horizontal line  $w=1$  at values of  $r \approx 3.6\%$  and  $r \approx 16.2\%$ .<sup>5</sup> Thus, switching and then reswitching between techniques takes place as the interest rate increases.

[Fig. 1 about here.]

Reswitching is a puzzling feature of the relationship between the wage ratio and interest rate embodied in Eq. (3). It is argued here that the relationship between  $w$  and  $r$  is subtler than it appears, the subtlety arising from the form of the function. Eq. (3) is a polynomial, therefore for each value of  $w$  there are twenty-five values of  $r$  that solve the equation. In general, an  $n^{\text{th}}$  order TVM polynomial has  $n$  solutions for  $r$ . Mathematically each value is as valid as any other. In order to explore the role played by every possible solution for the interest rate, an interim result is needed. This result enables a transformation of the wage equation to one in which all interest rates are not only visible, but functional.

### 3 The factorization of polynomials and Viete's formulas

Aleksandrov *et al.* (1969) summarize a well-known result about factorization of polynomials.

*'If we accept without proof the so-called fundamental theorem of algebra that every equation  $f(x)=0$ , where  $f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n$  is a polynomial in  $x$  of given degree  $n$  and the coefficients  $a_1, a_2, \dots, a_n$  are given real or complex numbers, has at least one real or complex root, and take into consideration that all computations with complex numbers are carried out with the same rules as with rational numbers, then it is easy to show that the polynomial  $f(x)$  can be represented (and in only one way) as a product of first-degree factors  $f(x) = (x - a)(x - b)\dots(x - l)$  where  $a, b, \dots, l$  are real or complex numbers.'*

Furthermore: *'Multiplying out the expression  $(x - a)(x - b)(x - c)\dots(x - l)$  and comparing the coefficients of the same powers of  $x$ , we see immediately that*

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<sup>5</sup> When the rate of interest is 25 percent the values for  $w_a$  and  $w_b$  obtained from Eqs. (1) and (2) are both zero which implies their ratio is undefined. However, Eq. (3) does define the ratio  $w$  when  $r = 25\%$ . The ratio at this point is 2.42; therefore, within the relevant range of interest rates, there are only two values of  $r$  when  $w$  is unity.

$$\begin{aligned}
-a_1 &= a + b + c + \dots + l, \\
a_2 &= ab + ac + \dots + kl, \\
-a_3 &= abc + abd + \dots, \\
&\dots\dots\dots \\
\pm a_n &= abc \dots l
\end{aligned}$$

which are Viète's formulas.' Aleksandrov *et al.* (1969)

These results, the factorization of a polynomial and the relationships between its parameters and its zeros, provide the route to a new wage function and its interpretation. It is shown that the wage ratio is expressible not only as a function of the interest rate and the parameters of Eq. (3), but also as a function of the interest rate and the zeros of Eq. (3). The latter expression is new and provides the reinterpretation of reswitching.

#### 4 Multiple interest rates in the Sraffa-Pasinetti example

Consider Eq. (3), which is in levels. Assume that the rate of interest,  $r$ , shifts to  $R$ . The wage ratio,  $w$ , becomes  $W$ . This is shown in Eq. (4), also in levels.

$$W = \frac{(1 + R)^{25} + 24}{20(1 + R)^8} \tag{4}$$

The next step introduces a third equation to bridge Eqs. (3) and (4), an equation in which  $(W-w)$  is a function of  $(R-r)$ , i.e., the new function is in differences instead of levels. It is this third equation that provides the new view of reswitching.

The result about the factorization of a polynomial is applied to Eq. (3) to produce Eq. (5). The values  $(1+r_i)$  in (5) are the zeros of Eq. (3).

$$(1+r)^{25} - 20w(1+r)^8 + 24 = [(1+r) - (1+r_1)][(1+r) - (1+r_2)] \dots [(1+r) - (1+r_{25})]$$

or, more succinctly,

$$(1+r)^{25} - 20w(1+r)^8 + 24 = (r-r_1)(r-r_2)\dots(r-r_{25}) \quad (5)$$

Substitute  $R$  for  $r$  in Eq. (5) to produce (6).

$$(1+R)^{25} - 20w(1+R)^8 + 24 = (R-r_1)(R-r_2)\dots(R-r_{25}) \quad (6)$$

Eq. (4) is rearranged and substituted into (6) to produce (7).

$$20W(1+R)^8 - 20w(1+R)^8 = \prod_1^{25} (R-r_i) \quad (7)$$

The last equation is rearranged into Eq. (8).

$$(W-w) = \frac{\prod_1^{25} (R-r_i)}{20(1+R)^8} \quad (8)$$

Eq. (8) is the required equation in differences that bridges Eqs. (3) and (4). There are many observations to be made about Eq. (8).

First, the left-hand side variable, the difference between the new and the old wage ratio, depends on the differences between the new interest rate,  $R$ , and all twenty-five initial values for  $r$  implied by the zeros of Eq. (3). One of the initial values of  $r$ , call it  $r_l$ , is designated the orthodox value that usually comes to mind when examining Eq. (3). Therefore the orthodox shift in the interest rate is  $(R-r_l)$ . By inspection, the relationship in Eq. (8), between the shift in the wage ratio  $(W-w)$  and the shift in the orthodox interest rate  $(R-r_l)$ , is more complicated than that suggested by a comparison of equations (3) and (4).

Secondly, it is possible to visualize the workings of Eq. (8). The  $i^{\text{th}}$  element  $(R-r_i)$  in Eq. (8) is the difference  $[(1+R)-(1+r_i)]$ . The set  $(1+r_i)$ , from  $i=1$



to  $n$ , consists of the zeros of Eq. (3). As will be shown, most of these zeros reside in the complex plane, off the real number line. If absolute values are taken on both sides of Eq. (8) it becomes (9).

$$|W - w| = \frac{\prod_{i=1}^{25} |R - r_i|}{20(1 + R)^8} \quad (9)$$

The elements  $|R - r_i|$  are distances in the complex plane. The distances are rays between the zeros of Eq. (3),  $(1 + r_i)$ , and the new interest rate,  $(1 + R)$ . The new rate is the locus of the set of rays. This situation is easily demonstrated using the numbers in the Sraffa-Pasinetti example.

In Pasinetti (1966) the prescribed range for the interest rate is 0% to 25% therefore a convenient value for the initial interest rate is zero. When  $r = r_i = 0$ , Eq. (3) implies  $w = 1.25$ . Knowing this value, Eq. (3) is solved for all twenty-five values of  $r$  that satisfy it. If these initial values for  $w$  and  $r_i$  are inserted into Eq. (9) then the remaining relationship is between  $R$  and  $W$ .

Fig. 2 is an Argand diagram showing all the roots of Eq. (3) when  $w = 1.25$ . Each ray joins a root to the locus. In the diagram, the locus is positioned arbitrarily at  $(1 + R) = 1.1$  ( $R = 10\%$ ). As the locus moves backwards or forwards along the real number line between 0% and 25%, the twenty-five rays change length. The product of these lengths affects the size of the overall product in the numerator of Eq. (9). In this way, Fig. 2 illustrates what is happening in Eq. (9) as  $R$  changes value and affects  $W$ .

[Fig. 2 about here.]

The third observation about the difference equation concerns the absolute values in Eq. (9). To understand how  $W$  behaves as  $R$  shifts, the signs (+/-) of some elements must be determined. Absolute values are released from some (but

not all) elements on both sides of Eq. (9) and correct signs are identified. For elements of the set  $r_i$  that lie off the real number line the absolute values of  $(R-r_i)$  are retained. For elements of the set that lie on the real number line the absolute values are removed and the signs of these ‘wholly real’ differences determined. The sign of the overall product is then apparent.

There are three real roots (interest rates) that satisfy the relevant equation:

$$1.25 = \frac{(1+r)^{25} + 24}{20(1+r)^8}. \text{ They are } -0.9907 \text{ (-199.07\%), } 1.0000 \text{ (0\%) and } 1.1891$$

(18.91%). As  $R$  passes through the range 0% to 25%, it lies consistently to the right of the first two values in the list, therefore their differences  $(R-r_i)$  are positive. The value of 18.91%, however, lies inside the range 0% to 25%. As  $R$  passes through the relevant range the difference is negative when  $R$  is to the left of 18.91% and positive when it is to the right. The sign of the overall product of differences varies accordingly. Eq. (9) is modified to Eq. (10) to reflect this situation in which  $r_i$ , from  $i = 1$  to 3, represents the three real values.

$$W - 1.25 = \left( \frac{1}{20} \right) \left( \frac{\prod_1^3 (R - r_i) \prod_4^{25} |R - r_i|}{(1 + R)^8} \right) \quad (10)$$

Table 1, Col. 1, contains values of  $R$  in the range 0% and 25%. Col. 2 contains the wage ratios over the range calculated in the orthodox manner from Eq. (4). Col. 3 contains the sign-adjusted product  $\prod_1^3 (R - r_i) \prod_4^{25} |R - r_i|$  as outlined in the previous paragraphs. Col. 4 contains the composite variable comprised of differences between rates, suitably discounted. Col. 5 contains the wage ratios determined by the new Eq. (10) as  $R$  varies from 0% to 25%. The route to the wage ratios in Col. 5 is different from the route to the wage ratios in Col. 2, yet the numbers are identical.

[Table 1 about here.]

The fourth observation about the new equation for differences in wage ratios is the most important. Given a single assumption about the initial value for  $r$  in Eq. (3), all other initial values for  $w$  and  $r_i$  are determined. When these initial values are inserted into (10) the resulting equation is the same as the levels equation (4) in one significant respect: inputting a given value for  $R$  into both equations yields the same value for  $W$ . The two equations, however, have entirely different structures. This fact has implications.

*‘A mathematical variable  $x$  is “something” or, more accurately, “anything” that may take on various numerical values.’ Aleksandrov (1969)*

In economics and finance, numerical values normally reside on the real number line; they are always relative to some fixed point on the line, usually zero. For example, in Eq. (4),  $R$  departs from 0% and moves along the real number line to 25%. There is one fixed point. It is zero. The value that varies is  $(R-0) = R$ ; therefore  $R$  is the variable.

Eq. (10) is different. Fig. 2 shows that  $R$  moves along the number line relative to twenty-five fixed points. Only three of the points are on the real number line and only one of them is zero. The remaining fixed points are distributed close to the unit circle in the complex plane. As  $R$  moves, twenty-five rays change length simultaneously, most of them at angles to the real number line. Faced with the structure of Eq. (10) it is difficult to maintain that the independent variable is  $(R-r_l)=(R-0)=R$  alone. The independent variable is better described by the composite variable comprising every element in which  $R$  appears:

$$\frac{\prod_1^3 (R-r_i) \prod_4^{25} |R-r_i|}{(1+R)^8}$$

The relationship between the wage ratio and the composite independent variable in Eq. (10) is monotonic because the structure of the equation is linear. From this perspective, reswitching does not occur in the Sraffa-Pasinetti example. The situation is illustrated in Fig. 3.

[Fig. 3 about here.]

In order to emphasize the last result, Eq. (10) is restated in another form. The orthodox increment in the interest rate is expressed as  $(R - r_1) = \Delta r$  which, because  $r_1 = 0$ , is equal to  $R$ .

$$W = 1.25 + \left( \frac{1}{20} \right) \left( \frac{\prod^3 (R - r_i) \prod^{25} |R - r_i|}{(1 + R)^8} \right) \Delta r .$$

The wage ratio on the left side of this equation is expressed as a function of the difference in the orthodox interest rate,  $\Delta r = R$ , on the far right. It is this relationship graphed in Fig. 1.

There is a problem. If Fig. 1 is to represent the true relationship between the wage ratio and the interest rate,  $R$ , then all the elements that stand between the two variables in Eq. (10) ought to be fixed parameters. In fact, most of the elements vary with  $R$ ; therefore they are components of the independent variable. It follows that the horizontal axis in Fig. 1 represents only one element of the independent variable; it does not represent the entire independent variable. Ignoring this situation gives the perception of reswitching.

## 5 Generalizing from the solution to the reswitching puzzle

Why resurrect the reswitching puzzle and offer a solution to it? The answer is that the solution offers a better understanding of the comparative statics of the TVM equation. Different inputs (interest rates) to the equation produce

different outputs (present or future values). The analysis in this article sheds new light on what happens in the TVM equation when an interest rate shifts. The TVM equation has many applications in economics and finance therefore it is worth considering the wider implications of the analysis.

A more general example of the TVM equation than the Sraffa-Pasinetti model is as follows. Eqs. (11) and (12) are equations for the present values of an arbitrary cash flow at two different interest rates. The interest rate shifts from  $r$  to  $R$  to produce the change in present value from  $p$  to  $P$ .

$$p = \sum_1^n \frac{c_i}{(1+r)^i} \quad (11)$$

$$P = \sum_1^n \frac{c_i}{(1+R)^i} \quad (12)$$

Eq. (13) is derived from Eqs. (11) and (12). It states that the relative shift in value is the discounted product of all shifts in the interest rate.

$$\left| \frac{\Delta p}{p} \right| = \frac{\prod |R - r_i|}{(1+R)^n} \quad (13)$$

The proof of Eq. (13) is not given here. As with the proof for the Sraffa-Pasinetti example above, it involves the factorization theorem.

The shifts from every  $r_i$  to  $R$  are given by the  $n$  entities:  $|R - r_i|$ . Alternatively, the shifts are expressed by the  $n$  mark-ups,  $m_i$ , implied by the relationship  $(1+R)(1+m_i) = (1+r_i)$ . This last relationship means Eq. (13) is transformed into the more compact expression (14) constructed of mark-ups in interest rates.

$$\left| \frac{\Delta p}{p} \right| = \prod |m_i| \quad (14)$$

In the transition from Eq. (11) to (12), the parameters (the cash flows,  $c_i$ ) remain constant. Viete's formulas demonstrate that the information embodied in the parameters (cash flows) of Eq. (11) is also found in the entire cluster of interest rates (zeros of Eq. (11)). This knowledge permits the construction of Eq. (13) in differences, or Eq. (14) in mark-ups. An outcome of the exercise is the realization that the independent variable is more complicated than it appears. The orthodox change in the interest rate,  $\Delta r = R - r_1$ , by itself, is not sufficient to explain the change in present value,  $\Delta p$ . There is activity going on in the complex plane that is not apparent from analysis confined to the real number line.

The structure of the equations can be adapted to the problem under investigation. Eqs. (3) and (9) are specific structures associated with the Sraffa-Pasinetti model. Eqs. (11) and (13) are more general structures. Whatever the structure, the general principle remains the same: the extent to which the dependent variable changes in response to a shift in the interest rate depends on all initial values of the interest rate before the shift. Ignoring this fact gives rise to puzzling relationships between variables. Reswitching is one example.

The analysis developed here, with its use of differences rather than levels, and the employment of an entire cluster of initial interest rates, holds implications for other topics in economics and finance. First, in the context of bond mathematics, Osborne (2005) produces a formula sought since Macaulay (1938): an algebraic formula for the interest elasticity of bond price that provides accurate results. The new formula has no need for convexity or the other terms of a Taylor series expansion. The formula employs the differences between the new interest rate and all initial interest rates. Secondly, in the context of capital budgeting, Osborne (2010) shows that net present value (NPV) per dollar invested is composed of the mark-downs of the cost of capital relative to all possible internal rates of return (IRR), thereby contributing to the debate about the relative merits

of NPV and IRR as investment criteria. This list of topics open to the ‘multiple-interest-rate’ approach cannot be complete; others are likely to exist.<sup>6</sup>

It follows that reswitching is a puzzle in a list of similar puzzles in economics and finance, each causing debate in its own field. The debates have a common factor: the TVM equation. One of the most useful questions that can be asked of the equation is how value varies under different assumptions about the interest rate. This deceptively simple question has proved difficult to answer satisfactorily. The long histories of reswitching and the other debates mentioned above are testament to the difficulty. Reswitching is an anomaly in the Kuhnian sense (Kuhn, 1962). It is one indicator among several of an issue with comparative static analysis that permeates economics and finance.

## 6 Conclusion

In this article the reswitching phenomenon is re-examined in the context of the Sraffa-Pasinetti model. The phenomenon does not occur when it is analyzed using a new TVM equation expressed in differences rather than levels, an equation containing all possible shifts in interest rates, rather than the single, orthodox shift alone. The methodology can be generalized: it applies to a TVM polynomial of any order; and it applies to topics other than reswitching.

Questions remain. First, as mentioned in the introduction, can the new approach be adapted from comparative static analysis at a moment in time to the analysis of a process through time? Secondly, what are the implications of the

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<sup>6</sup> Dorfman (1981) is possibly the earliest example; Dorfman’s mode of analysis, however, is different from that adopted here. Dorfman uses all interest rates in their ‘raw’, complex form, whereas the approach described here uses absolute differences between interest rates, which are real numbers. The relationship between the two modes of analysis is an open question and is left for future research.

The ‘multiple interest rate’ literature in the context of capital budgeting is summarized in Magni (2010). Most authors discuss only the multiple real rates. Hazen (2003) and Pierru (2010) are recent exceptions in which there is explicit use of complex rates. Their discussion is of rates *per se*, used individually as investment criteria (internal rates of return); they do not use the rates simultaneously as ingredients in a formula, as in Dorfman’s article or this one.

analysis described in this work for the capital controversies overall? There was more to the Cambridge capital controversies than the reswitching puzzle. Thirdly, as noted above, what are the implications for other topics in economics and finance that employ the TVM equation? Finally, a better understanding of reswitching comes at the price of a deeper question: what economic or financial meaning can be attributed to all possible solutions for the interest rate, especially the complex? Answers to these questions are left for future research. In the meantime, this article provides a different perspective on a famous debate.

Pasinetti (2003) is a response to the recent review of the Cambridge capital controversies by Cohen and Harcourt (2003). Pasinetti states that ...

*'... one fact remains undisputed as a result of the 1966 QJE symposium, namely that the relationship between capital ... and its "factor price" is in general a nonmonotonic relation. This characteristic is contrary to the assumptions underlying neoclassical capital theory ...'*  
(Pasinetti, 2003, pp. 227-228)

The full import for neoclassical theory of the 'multiple-interest-rate' analysis offered in this article has yet to be worked out; but in one respect at least there is a clear implication: when viewed from the two dimensions of the complex plane, there is no reswitching in the Sraffa-Pasinetti model described in Pasinetti (1966).

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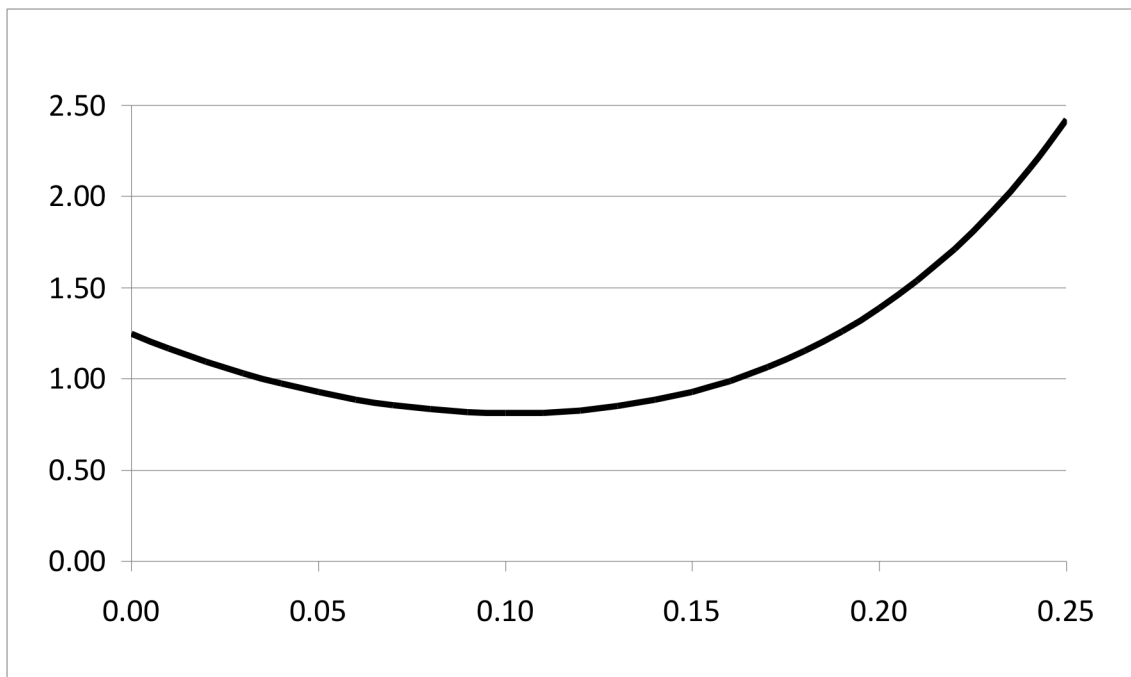
**Table 1. The numbers of the Sraffa-Pasinetti example**

The wage ratios in Col. 2 and the numbers in Col. 5 are identical yet they are calculated in two different ways. Col. 2 contains values for  $w_a/w_b$  produced by Eq. (4). Col. 5 is calculated from Eq. (10) using the sign-adjusted products of the differences between interest rates. Note the change of sign in Col. 3 that occurs as the interest rate,  $R$ , passes by one of the three real roots at 18.91 percent. The causal variable (the composite variable containing  $R$ ) is in Col. 4; it is the variable on the x-axis of Fig. 3.

Col. 1	Col. 2	Col. 3	Col. 4	Col. 5
$R$	$w_a / w_b$	$\prod_1^3 (R - r_i) \prod_4^{25}  R - r_i $	$\frac{\prod_1^3 (R - r_i) \prod_4^{25}  R - r_i }{(1 + R)^8}$	$1.25 + \left(\frac{1}{20}\right) * \text{Col.4}$
0	1.2500	0.0000	0.0000	1.2500
0.01	1.1674	-1.7890	-1.6521	1.1674
0.02	1.0942	-3.6509	-3.1160	1.0942
0.03	1.0299	-5.5755	-4.4013	1.0299
0.04	0.9742	-7.5484	-5.5155	0.9742
0.05	0.9268	-9.5500	-6.4638	0.9268
0.06	0.8875	-11.5543	-7.2493	0.8875
0.07	0.8564	-13.5272	-7.8730	0.8564
0.08	0.8333	-15.4248	-8.3335	0.8333
0.09	0.8186	-17.1910	-8.6276	0.8186
0.10	0.8125	-18.7550	-8.7494	0.8125
0.11	0.8155	-20.0280	-8.6907	0.8155
0.12	0.8280	-20.8990	-8.4408	0.8280
0.13	0.8507	-21.2306	-7.9861	0.8507
0.14	0.8845	-20.8527	-7.3101	0.8845
0.15	0.9303	-19.5566	-6.3931	0.9303
0.16	0.9894	-17.0861	-5.2117	0.9894
0.17	1.0631	-13.1285	-3.7388	1.0631
0.18	1.1529	-7.3029	-1.9428	1.1529
0.19	1.2606	0.8534	0.2122	1.2606
0.20	1.3884	11.9008	2.7677	1.3884
0.21	1.5385	26.5165	5.7708	1.5385
0.22	1.7137	45.5174	9.2747	1.7137
0.23	1.9170	69.8865	13.3399	1.9170
0.24	2.1517	100.8043	18.0346	2.1517
0.25	2.4218	139.6862	23.4355	2.4218

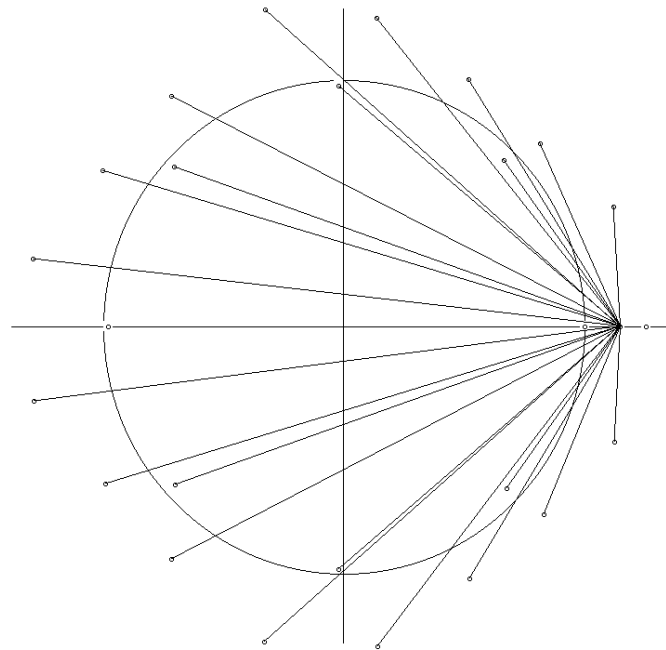
**Fig. 1. The Sraffa-Pasinetti example**

The wage ratio  $w=w_a/w_b$  on the y-axis varies with the rate of interest on the x-axis. The wage ratio is equal to unity at two values of the rate of interest: 3.6 percent and 16.2 percent; therefore reswitching is apparent.



**Fig. 2. An Argand diagram illustrating the Sraffa-Pasinetti example**

All twenty-five roots of Eq. 3, i.e.,  $1.25 = \frac{(1+r)^{25} + 24}{20(1+r)^8}$ , are plotted in the complex plane. Twenty-two are in eleven, conjugate complex pairs and three are real at  $-0.9907$ ,  $1.000$  and  $1.1891$ . The rays stretch between each of the roots,  $(1+r_i)$ , and the locus  $(1+R)$ . The locus in the figure is arbitrarily set at  $1.1$ . As  $(1+R)$  moves back and forth along the real number line between  $1.00$  and  $1.25$ , the rays change length and their product changes value, thereby affecting the value of the numerator on the right-hand side of Eq. (9). The unit circle is shown to provide scale.



**Fig. 3. The Sraffa-Pasinetti example reinterpreted**

The wage ratio  $w_a/w_b$  on the y-axis varies with the composite variable containing differences between interest rates on the x-axis. The graph is based on Eq. (10) in the text. The causal variable is in Col. 4 of Table 1. The wage ratio is equal to unity at only one value of the causal variable because the relationship is linear; there is no reswitching.

