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Stability issues in Kaleckian models driven by autonomous demand growth – Harrodian instability and debt dynamics

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Stability issues in Kaleckian models driven by autonomous demand growth – Harrodian instability and debt dynamics[*]

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Abstract
Sraffian supermultiplier models, as well as Kaleckian distribution and growth models making use of non-capacity creating autonomous demand growth in order to cope with Harrodian instability, have paid little attention to the financial side of autonomous demand growth as the driver of the system. Therefore, we link the issue of Harrodian instability in Kaleckian models driven by non-capacity creating autonomous demand growth with the associated financial dynamics. For a simple model with autonomous government expenditure growth, zero interest rates and no consumption out of wealth, we find that adding debt dynamics does not change the results obtained by Lavoie (2016) for a model without debt, i.e. the long-run equilibrium is stable if Harrodian instability is not too strong and the autonomous growth rate does not exceed a maximum given by the long-run equilibrium saving rate. Introducing interest payments on government debt as well as consumption out of wealth into the model, however, changes the stability requirements: First, the autonomous growth rate of government expenditures should not fall short of the exogenous monetary interest rate. Second, this growth rate should not exceed a maximum given by the saving rate in long-run equilibrium minus the propensity to consume out of wealth. Third, Harrodian instability may be stronger than in the simple model without violating long-run overall stability, in particular, if the rate of interest is very low and the growth rate of government expenditures is close to the mentioned upper limit. We claim that, irrespective of the relevance or irrelevance of Harrodian instability, it is necessary to introduce financial variables into models driven by non-capacity creating autonomous demand in order to assess the long-run (in-)stability and sustainability of growth.

Keywords: Supermultiplier, autonomous demand growth, Kaleckian models, Harrodian instability, financial (in)stability

JEL code: E11, E12, E25, E62

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1. Introduction

Distribution and growth models driven by a non-capacity creating component of aggregate demand have become increasingly popular in heterodox distribution and growth theories, recently generating a whole special issue in *Metroeconomica*, 2019, 70 (2). Initially a 'Sraffian supermultiplier' model driven by autonomous demand was proposed by Serrano (1995a, 1995b), and has further been discussed, developed and applied by Cesaratto (2015), Cesaratto/Serrano/Stirati (2003), Dejuan (2005), Fazzari et al. (2013), Fazzari/Ferri/Variato (2020), Freitas/Serrano (2015, 2017), Girardi/Pariboni (2016), Pariboni (2016), among others.

Starting with Allain (2015) and Lavoie (2016), also several Kaleckian authors have turned towards introducing a Sraffian supermultiplier process into their models of distribution and growth in order to defend this approach against the Harrodian and Marxian critique. These critics had argued that the Kaleckian notion of an endogenous rate of capacity utilisation beyond the short run is not sustainable, that Kaleckian models are thus facing the problem of Harrodian instability, and that the Kaleckian results of the paradox of saving and a potential paradox of costs cannot be validated beyond the short run.[1] Introducing an autonomous growth rate of a non-capacity creating component of aggregate demand, Kaleckian authors have shown in basic and more elaborated models, which allow for convergence towards a normal or target rate of capacity utilisation in the long run, first, that under some weak conditions autonomous demand growth is able to tame Harrodian instability and, second, that the paradox of saving and a potential paradox of costs can be preserved for the long-run growth path (Allain 2015, 2019, Dutt 2019, 2020, Lavoie 2016, Nah/Lavoie 2017, 2018, 2019a, 2019b, Palley 2019). In these models, the autonomous growth rate of a non-capacity creating component of aggregate demand, i.e. autonomous consumption, residential investment, exports or government expenditures, determines long-run growth, and, under the conditions that Harrodian instability in the investment function is not too strong, provides for a stable adjustment towards the normal rate of capacity utilisation in the long run. In those models, a change in the propensity to save or in the profit share will have no effect on the long-run growth rate, but will affect the traverse and thus the long-run growth path. The paradox of saving and the possibility of a paradox of costs from the short run thus disappear with respect to the long-run growth rate, but they remain valid with respect to the long-run growth path.

Of course, the Sraffian supermultiplier models and the integration of autonomous demand growth into Kaleckian models have been critically discussed. This critique has addressed the implied full endogeneity of investment with respective to output growth, i.e. fully induced investment. Furthermore, in particular, the assumption of any expenditure growth being fully autonomous of the variation of income and output in the long run, for which these models have been designed, and the exclusion of financial instability issues have been targeted (Nikiforos 2018, Skott 2019).

Our contribution is related to the question of long-run autonomy of components of demand from income/output and the issue of financial instability. Not only from a post-Keynesian perspective should it be evident that this autonomy implies that demand can be

[1] For the Harrodian/Marxian critique see, for example, Dumenil/Levy (1999), Shaikh (2009) and Skott (2010, 2012). For a review of this critque and Kaleckian respones see Hein/Lavoie/van Treeck (2011, 2012).

financed independently of income generated in the long run. This is only possible if those sectors generating autonomous demand growth have wealth they can draw on and/or access to credit—the private household sector in the case of autonomous consumption demand and residential investment, the external sector (i.e. the importing countries) in case of exports, and the government in the case of government expenditures as the growth driver. Empirically, financial dynamics are most important when it comes to the sustainability of autonomous demand driven growth, as is conceded by some of the proponents of this approach (Fiebiger 2018, Fiebiger/Lavoie 2019).

Interestingly, however, the dynamics of financial assets and liabilities—and of debt in particular—associated with autonomous demand growth have hardly been explored so far. A systematic examination of the potential limits generated from the monetary and financial side to the sustainability of autonomous demand growth would thus seem desirable. Pariboni (2016) has introduced household debt into a supermultiplier growth model driven by autonomous consumption, but has not carefully studied the dynamic interaction of Harrodian instability and debt dynamics. Brochier/Macedo e Silva (2019) have included financial wealth/liabilities into a supermultiplier stock-flow consistent model driven by autonomous consumption growth, but have only numerically simulated the respective dynamics. Hein (2018) has included autonomous government expenditure growth financed by credit in a Kaleckian distribution and growth model, and has studied the dynamics of government deficits and government debt. However, he has maintained the Kaleckian assumption of an endogenous rate of capacity utilisation and has thus not addressed the Harrodian instability issue. Dutt (2020), on the contrary, has included the attainment of a long-run normal rate of utilisation into a model driven by autonomous government expenditure growth financed by credit and has examined the related debt dynamics. But he has 'switched off' Harrodian instability, assuming firms' expectation about long-run growth are given by the growth rate of government expenditures.[2]

Our current contribution attempts to link the issue of Harrodian instability in Kaleckian models driven by non-capacity creating autonomous demand growth with the associated financial dynamics. We will do this in two steps, building on Lavoie's (2016) basic model, in the first step, and on Hein's (2018) model in the second step. In the basic model, however, we will replace autonomous consumption growth proposed by Lavoie (2016) by autonomous government expenditure growth as the driver of growth for two reasons. First, we feel that the 2007-9 financial and economic crises, as well as the following Eurozone crisis, have shown the limits of credit-financed autonomous consumption, residential investment and export growth as autonomous long-run drivers of growth, such that government expenditure growth remains the 'realistic' alternative, under certain circumstances. Second, using government expenditures as autonomous growth driver allows for a helpful simplification of zero interest rates and neglect of debt repayment. In Section 2 we will thus provide an extremely simple model in which we study Harrodian instability and debt dynamics, ignoring both interest payments and consumption out of wealth. In order to highlight the role of debt dynamics, we

[2] This assumption can already be found in Serrano (1995, p. 77), as Marc Lavoie has pointed out to us.

will even 'switch off' Harrodian instability in the first version, but not the notion that firms operate at a normal or target rate of utilisation in the long run, following the procedure suggested by Dutt (2019, 2020). Section 3 will then include a positive interest rate on government debt as well as consumption out of financial wealth, as in Hein (2018). We will start with a model version without Harrodian instability, but with a normal rate of utilisation obtained in the long run, and then we will add a Harrodian instability equation in the second version. In the final Section 4 we will summarise and conclude. Before moving to Section 2 we should stress that the purpose of our paper is conceptual and didactic: We would like to add to the understanding of the (potentially destabilising) role of financial dynamics in models driven by autonomous non-capacity creating expenditures, with or without Harrodian instability. The extremely simple models should thus not be taken to the data without further refinements.[3]

2. Harrodian instability and debt dynamics: a simple version based on Lavoie (2016)

We assume a closed economy, in which a single good for investment and consumption purposes is produced by a fixed coefficient technology, using a non-depreciating capital stock (K) and direct labour (L). For the latter there is no supply constraint. The rate of utilisation of productive capacities is defined as the ratio of output to the capital stock: $u = Y/K$. Income is distributed between capitalists and workers, and the profit share ($h = \pi/Y$) is determined by mark-up pricing of firms in an oligopolistic goods market, with the mark-up being affected by the degree of price competition in the goods market and the bargaining power of workers in the labour market. With given institutional conditions, prices are constant, so that nominal and real variables coincide at a price level $p = 1$. Workers do not save, and only capitalists save a fraction of their profits determined by the propensity to save out of profit income s_Π. For the sake of simplicity, we assume that all profits are distributed to the capitalists' households as the owners of the firm. We normalise saving (S) by the capital stock and get the saving rate (σ):

$$\sigma = \frac{S}{K} = s_\pi \frac{\pi}{Y}\frac{Y}{K} = s_\pi hu, \qquad s_\pi > 0 \tag{1}$$

Capitalists decide to invest (I) according to the expected trend rate of growth (α). Whenever the actual rate of capacity utilisation (u) falls short of (exceeds) the target or normal rate of utilisation (u_n), they slow down (accelerate) the rate of capital accumulation (g):

$$g = \frac{I}{K} = \alpha + \beta(u - u_n), \qquad \beta > 0 \tag{2}$$

Government consumption expenditure (G) drives our model economy and grows with the rate γ. Governments finance their expenditures by issuing bonds which are held by the capitalists. In this version of the model we ignore interest rates, assuming that bonds are issued at a zero

[3] For different views on this and other issues, see Lavoie (2017) and Skott (2017, 2019).

rate or that governments emit money. In addition, wealth effects on capitalists' consumption are ignored for now. Government expenditures are equal to the government deficit and are also normalised by the capital stock such that for the government expenditures-capital stock ratio (*b*) we get:

$$b = \frac{G}{K} = \frac{G_0 e^{\gamma t}}{K}, \qquad \gamma > 0 \tag{3}$$

The short-run goods market equilibrium is given by:

$$\sigma = g + b \tag{4}$$

and the stability condition by:

$$s_\pi h - \beta > 0 \tag{5}$$

Firms adjust output to demand in the short run by means of varying the rate of capacity utilisation. From equation (1), (2), (3) and (4) we thus obtain the short-run equilibrium rate of capacity utilisation (u^*) for a given government expenditures-capital ratio and a given government debt-capital ratio ($\lambda = L/K$) inherited from the past:

$$u^* = \frac{\alpha - \beta u_n + b}{s_\pi h - \beta} \tag{6}$$

In the long run, government expenditures grow with the rate γ, and the government expenditures-capital ratio changes according to:[4]

$$\hat{b} = (\gamma - g) = \gamma - \alpha - \beta(u^* - u_n) \tag{7}$$

with variables with a hat denoting growth rates. Likewise, in the long run we have an endogenous rate of change of the government debt-capital ratio:

$$\hat{\lambda} = \frac{b}{\lambda} - g = \frac{b}{\lambda} - \alpha - \beta(u^* - u_n) \tag{8}$$

[4] We have chosen to formulate the dynamic equations in growth rates, in order to compare our results with those by Lavoie (2016), as will be seen below. Doing that we are also following the procedures in Dutt (2019), Nah/Lavoie (2017, 2018, 2019a, 2019b), and several others. For a discussion about the consequences of using growth rates instead of time rates of change, see, for example Skott (2017) and the reply by Lavoie (2017).

Before we consider Harrodian instability and its effects, we begin by assuming with Dutt (2019, 2020) that firms may have 'rational'—or better 'reasonable'—expectations about the trend rate of growth given by government expenditure growth γ.[5]

$$\alpha = \gamma \tag{9}$$

Hence, for the time being and for the purposes of comparison, we begin with our simplest model where Harrodian instability is 'switched off'.

To analyse the stability of long-run equilibrium values, we must first determine our long-run equilibrium ($**$), setting each of our dynamic equations equal to zero and making use of the short-run equilibrium rate of capacity utilisation from equation (6):

$$u^{**} = u_n \tag{10}$$

$$g^{**} = \alpha = \gamma \tag{11}$$

$$b^{**} = s_\pi h u_n - \gamma \tag{12}$$

$$\lambda^{**} = \frac{b^{**}}{g^{**}} = \frac{s_\pi h u_n - \gamma}{\gamma} \tag{13}$$

Since α does not change endogenously, because we have 'switched-off' Harrodian instability, there will only be two dynamic equations in the long run, equation (7) for the growth rate of the government expenditures-capital ratio and equation (8) for the government debt-capital ratio. The corresponding Jacobian matrix is given in equation (14a) and is evaluated at the long-run equilibrium values in (14b):

$$J = \begin{bmatrix} \dfrac{\partial \hat{b}}{\partial b} & \dfrac{\partial \hat{b}}{\partial \lambda} \\ \\ \dfrac{\partial \hat{\lambda}}{\partial b} & \dfrac{\partial \hat{\lambda}}{\partial \lambda} \end{bmatrix} \tag{14a}$$

$$J^{**} = \begin{bmatrix} \dfrac{-\beta}{s_\pi h - \beta} & 0 \\ \\ \dfrac{s_\pi h(\gamma - \beta u_n)}{(s_\pi h u_n - \gamma)(s_\pi h - \beta)} & \dfrac{-\gamma^2}{(s_\pi h u_n - \gamma)} \end{bmatrix} \tag{14b}$$

[5] To quote Dutt (2019, FN 5): 'Assuming "rational" expectations does not require that firms know the entire structure of the model as in new classical macroeconomic models, but only that the expected long-run growth is equal to the known rate of growth of the exogenous component …'.

For local stability in this 2x2 system, the trace of the Jacobian must be negative and the determinant must be nonnegative. The trace and the determinant are given in equations (15) and (16) respectively.

$$Tr(J^{**}) = \frac{-\beta(s_\pi h u_n - \gamma) - \gamma^2(s_\pi h - \beta)}{(s_\pi h u_n - \gamma)(s_\pi h - \beta)} \tag{15}$$

$$Det(J^{**}) = \frac{\beta\gamma^2}{(s_\pi h u_n - \gamma)(s_\pi h - \beta)} \tag{16}$$

For $Det(J^{**}) \geq 0$, it must be the case that $s_\pi h u_n - \gamma > 0$. If this is met, then we have $Tr(J^{**}) < 0$, which means that positive long-run equilibrium values of the government expenditures- and the government debt-capital ratios are stable. However, if the autonomous growth rate of government expenditure becomes too high, such that $s_\pi h u_n - \gamma < 0$, the long-run equilibrium values for the government deficit- and debt-capital ratios will be negative and also turn unstable, even without Harrodian instability. The inclusion of deficit and debt dynamics thus imposes an upper limit on the growth rate of autonomous government expenditures for stability purposes. This limit is given by the saving rate at normal capacity utilisation: $\gamma < s_\pi h u_n$.

Let us now consider the same model, but with Harrodian instability, as formulated by by Lavoie (2016), 'switched on'. Firms' assessment of the trend rate of growth will now also change, if the goods market equilibrium rate of capacity utilisation deviates (persistently) from the target or normal rate of utilisation. We thus get the following Harrodian equation, which replaces equation (9), with μ denoting the Harrodian instability parameter:

$$\hat{\alpha} = \mu\beta(u^* - u_n), \quad \mu > 0 \tag{17}$$

Our simple model thus now has three dynamic equations in (7), (8) and (17), to be examined for long-run equilibrium stability. Our long-run equilibrium values are the same as before, given by equations (10) – (13).To analyse the stability of the long-run equilibrium, we find the Jacobian matrix (18a) of the dynamic system in equations (7), (8) and (17), which is then evaluated at the long-run equilibrium (18b):

$$J = \begin{bmatrix} \frac{\partial \hat{b}}{\partial b} & \frac{\partial \hat{b}}{\partial \alpha} & \frac{\partial \hat{b}}{\partial \lambda} \\ \frac{\partial \hat{\alpha}}{\partial b} & \frac{\partial \hat{\alpha}}{\partial \alpha} & \frac{\partial \hat{\alpha}}{\partial \lambda} \\ \frac{\partial \hat{\lambda}}{\partial b} & \frac{\partial \hat{\lambda}}{\partial \alpha} & \frac{\partial \hat{\lambda}}{\partial \lambda} \end{bmatrix} \tag{18a}$$

$$J^{**} = \begin{bmatrix} \frac{-\beta}{s_\pi h - \beta} & \frac{-s_\pi h}{s_\pi h - \beta} & 0 \\ \frac{\mu\beta}{s_\pi h - \beta} & \frac{\mu\beta}{s_\pi h - \beta} & 0 \\ \frac{s_\pi h(\gamma - \beta u_n)}{(s_\pi h u_n - \gamma)(s_\pi h - \beta)} & \frac{-s_\pi h}{s_\pi h - \beta} & \frac{-\gamma^2}{s_\pi h u_n - \gamma} \end{bmatrix} \tag{18b}$$

For local stability of this 3x3 system, the following Routh-Hurwitz (R-H) conditions must hold:

1. $Det(J^{**}) < 0$
2. $Tr(J^{**}) < 0$
3. $Det(J_1^{**}) + Det(J_2^{**}) + Det(J_3^{**}) > 0$
4. $-Tr(J^{**})[Det(J_1^{**}) + Det(J_2^{**}) + Det(J_3^{**})] + Det(J^{**}) > 0$

In order to check these four conditions for our dynamic system, we obtain from the Jacobian matrix in equation (18b) the following results, with $Det(J_1^{**}), Det(J_2^{**}), Det(J_3^{**})$ as the determinants of the three second-order principal sub matrices:

$$Det(J^{**}) = \frac{-\mu\beta\gamma^2}{(s_\pi h u_n - \gamma)(s_\pi h - \beta)} < 0 \tag{19}$$

$$Tr(J^{**}) = \frac{-\beta(1-\mu)(s_\pi h u_n - \gamma) - \gamma^2(s_\pi h - \beta)}{(s_\pi h u_n - \gamma)(s_\pi h - \beta)} < 0 \tag{20}$$

$$Det(J_1^{**}) + Det(J_2^{**}) + Det(J_3^{**}) = \frac{(1-\mu)\beta\gamma^2 + \mu\beta(s_\pi h u_n - \gamma)}{(s_\pi h u_n - \gamma)(s_\pi h - \beta)} > 0 \tag{21}$$

$$-Tr(J^{**})[Det(J_1^{**}) + Det(J_2^{**}) + Det(J_3^{**})] + Det(J^{**})$$

$$= \frac{(1-\mu)\beta\,[\gamma^4(s_\pi h - \beta) + \beta\gamma^2(1-\mu)(s_\pi h u_n - \gamma) + \beta\mu(s_\pi h u_n - \gamma)^2]}{(s_\pi h u_n - \gamma)^2(s_\pi h - \beta)^2} \tag{22}$$

$$> 0$$

We will only discuss stability properties for economically meaningful (i.e. non-negative) long-run equilibrium values of the government expenditures-capital ratio and the government debt-capital ratio. For these positive long-run equilibrium values, from equations (12) and (13) we therefore need $s_\pi h u_n - \gamma > 0$. For the four R-H conditions, we obtain:

1. Since we assume that the stability condition for the goods market equilibrium (5), $s_\pi h - \beta > 0$, is met, the first R-H stability condition of a negative determinant will be

fulfilled for positive long-run equilibrium values of the government expenditures-capital ratio and the government debt-capital ratio, as is clear from equation (19).

2. To fulfil the second condition of a negative trace, from equation (20) we need $\mu < 1 + \frac{\gamma^2(s_\pi h-\beta)}{\beta(s_\pi hu_n-\gamma)}$.
3. According to equations (21), the third R-H condition of a positive sum of the determinants of the three second-order principal sub matrices, in the case $\gamma^2 > (s_\pi hu_n - \gamma)$ will be met), if $\mu < \frac{\gamma^2}{\gamma^2-(s_\pi hu_n-\gamma)}$. Note that this is limit on μ is positive and strictly larger than one. In the case $\gamma^2 < (s_\pi hu_n - \gamma)$, there is no restriction on μ implied by this third Routh-Hurwitz condition.
4. The fourth R-H condition is always fulfilled, if $\mu < 1$, as can be seen in equation (22). In general, the sign of the term in equation (22) is determined by the quadratic in the numerator, whose roots are at $\mu = 1$ and $\mu = \frac{\gamma^2}{\gamma^2-(s_\pi hu_n-\gamma)}\left[1 + \frac{(s_\pi h-\beta)\gamma^2}{\beta(s_\pi hu_n-\gamma)}\right]$. If $\gamma^2 < (s_\pi hu_n - \gamma)$, this quadratic is concave down and the second root is negative, so that the values of μ that satisfy this fourth R-H condition are strictly lower than one. If, on the other hand, $\gamma^2 > (s_\pi hu_n - \gamma)$, then the quadratic is convex up and the second root is positive and greater than one. Hence, also in this case, $\mu < 1$ satisfies this fourth R-H condition, but so too does $\mu > \frac{\gamma^2}{\gamma^2-(s_\pi hu_n-\gamma)}\left[1 + \frac{(s_\pi h-\beta)\gamma^2}{\beta(s_\pi hu_n-\gamma)}\right]$. However, a value of μ greater than this second root logically conflicts with the restriction imposed by R-H condition three. Hence, such a high value of μ is impossible for overall stability of the system, and the R-H conditions are collectively satisfied only for $\mu < 1$.

Summing up the conditions for the existence and stability of economically meaningful equilibria, first, from stability condition one, we need that the autonomous growth rate of government expenditures is below the saving rate at normal capacity utilisation, $s_\pi hu_n > \gamma$. This is the same as in the model without Harrodian instability. Second, from stability conditions two to four, we obtain that the system will be stable, if the Harrodian instability parameter is below one, i.e. $\mu < 1$. The model with deficit and debt dynamics is thus in line with the results of Lavoie (2016), who had provided the same upper limit for the growth rate of autonomous expenditures and also argued that taming Harrodian instability requires $\mu < 1$.

3. Government deficit and debt dynamics in a model with interest income and consumption out of financial and real wealth

Including interest payments on government debt and consumption out of real and financial wealth into the model, we follow Hein (2018). Distinguishing between the firm sector and the rentiers' household sector, we assume for simplicity that long-term finance of the real capital stock only consists of equity issued by the firms and held by the rentiers (*K* = *E*). Firms distribute all profits as dividends to the shareholders, i.e. there are no retained earnings. Therefore, rentiers hold the equity issued by the firms, the value of which is equal to the capital stock, and the debt issued by the government. They receive all the profits generated in the production sector (*hY*) and the interest paid out by the government (*iL*), the latter determined by the exogenous rate of interest and the stock of government debt. They consume part of their current profit and interest income and save the rest according to their propensity to save (s_R). Furthermore, we assume that rentiers consume part of their wealth according to their propensity to consume out of wealth (c_{WR}). Consumption out of wealth thus reduces saving out of rentiers' current income and our saving function turns to:

$$\sigma = \frac{S}{K} = \frac{s_R(hY + iL)}{K} - \frac{c_{WR}(L + K)}{K} = s_R(hu + i\lambda) - c_{WR}(\lambda + 1), \tag{23}$$

$$s_R > 0, c_{WR} \geq 0$$

From Section 2, we keep the investment function (2) and the function (3) for government expenditures for goods and services, i.e. government consumption, which rise again with the autonomous growth rate γ.

$$g = \frac{I}{K} = \alpha + \beta(u - u_n), \quad \beta > 0 \tag{2}$$

$$b = \frac{G}{K} = \frac{G_0 e^{\gamma t}}{K}, \quad \gamma > 0 \tag{3}$$

Government expenditures (G) are now equivalent to the primary government deficit. For the total deficit we have to add government interest payments. The short-run equilibrium condition turns to:

$$\sigma = g + b + i\lambda \tag{24}$$

and the stability condition for the goods market equilibrium becomes:

$$s_R h - \beta > 0 \tag{25}$$

Equations (23), (2), (3) and (24) yield the short-run equilibrium rate of capacity utilisation:

$$u^* = \frac{\alpha - \beta u_n + b + c_{WR} + \lambda[i(1 - s_R) + c_{WR}]}{s_R h - \beta} \tag{26}$$

In order to analyse the long-run dynamic properties related to the government primary deficit- and government debt-capital ratios in our model, and to compare it to the first version of the model from the previous section, we begin again with Harrodian instability 'switched off', as in equation (9):

$$\alpha = \gamma \tag{9}$$

We thus only obtain two dynamic equations, one for the government expenditures/primary government deficit-capital ratio and one for the government debt-capital ratio:[6]

$$\hat{b} = \gamma - g = \gamma - \alpha - \beta(u^* - u_n) \tag{27}$$

$$\hat{\lambda} = \frac{b + \lambda(i - g)}{\lambda} = \frac{b}{\lambda} + i - \alpha - \beta(u^* - u_n) \tag{28}$$

Setting equations (27) and (28) equal to zero and using equations (9) and (26), we have for the long-run equilibrium:

$$u^{**} = u_n \tag{29}$$

$$g^{**} = \gamma \tag{30}$$

$$\lambda^{**} = \frac{s_R h u_n - \gamma - c_{WR}}{\gamma + c_{WR} - s_R i} \tag{31}$$

$$b^{**} = \frac{(\gamma - i)(s_R h u_n - \gamma - c_{WR})}{\gamma + c_{WR} - s_R i} \tag{32}$$

In order to analyse the long-run stability properties, we define:

- $A = s_R h - \beta$,
- $B = i(1 - s_R) + c_{WR}$,
- $C = c_{WR} + \gamma - s_R i$,
- $D = s_R h u_n - \gamma - c_{WR}$,
- $F = \gamma - i$.

Based on our assumptions, we know:

- $A > 0$, because of the assumption of goods market equilibrium stability,
- $B > 0$, because $s_R < 1$,
- $F < C$, because $s_R < 1$ and $c_{WR} > 0$,
- $F + B = C$.

[6] Different from Hein (2018), we have the dynamic equations again in growth rates, whereas Hein (2018) has time rates of change. See also footnote 3 above.

The Jacobian matrix of the two dimensional dynamics from equations (27) and (28), taking into account the goods market equilibrium rate of capacity utilisation from equation (26), to be evaluated at the long-run equilibrium is given by:

$$J = \begin{bmatrix} \frac{\partial \hat{b}}{\partial b} & \frac{\partial \hat{b}}{\partial \lambda} \\ \\ \frac{\partial \hat{\lambda}}{\partial b} & \frac{\partial \hat{\lambda}}{\partial \lambda} \end{bmatrix} \tag{33a}$$

$$J^{**} = \begin{bmatrix} \frac{-\beta}{A} & \frac{-\beta B}{A} \\ \\ \frac{AC - \beta D}{AD} & \frac{-ACF - \beta BD}{AD} \end{bmatrix} \tag{33b}$$

From this, we obtain for the trace and the determinant:

$$Tr(J^{**}) = \frac{-\beta D(1 + B) - AFC}{AD}$$

$$= \frac{-\beta(s_R h u_n - \gamma - c_{WR})[1 + i(1 - s_R) + c_{WR}] - (\gamma - i)(s_R h - \beta)(c_{WR} + \gamma - s_R}{(s_R h - \beta)(s_R h u_n - \gamma - c_{WR})} \tag{34}$$

$$Det(J^{**}) = \frac{\beta C^2}{AD} = \frac{\beta(c_{WR} + \gamma - s_R i)^2}{(s_R h - \beta)(s_R h u_n - \gamma - c_{WR})} \tag{35}$$

The potential cases for our model are shown in Table 1. They only depend on the signs of three terms, $D = s_R h u_n - \gamma - c_{WR}$, $F = \gamma - i$ and $C = c_{WR} + \gamma - s_R i$. We have already mentioned the signs for the other terms based on our model assumptions above. From the trace and the determinant, we now also get that

- $s_R h u_n - \gamma - c_{WR} = D \neq 0$,

because otherwise the trace and the determinant in equations (34) and (35) would not be defined. Furthermore, we need:

- $\gamma + c_{WR} - s_R i = C \neq 0$,

because otherwise b^{**} and λ^{**} would not be defined, as can be seen in equations (31) and (32).

Table 1: Stability of the long-run equilibrium values of government primary deficit-capital ratio and government debt-capital ratio without Harrodian instability								
Case	$D = s_R h u_n - \gamma - c_{WR}$	$F = \gamma - i$	$C = c_{WR} + \gamma - s_R i$	$\frac{DF}{C} = b^{**}$	$\frac{D}{C} = \lambda^{**}$	$Tr(J^{**})$	$Det(J^{**})$	Stability
1	***+***	***+***	***+***	***+***	***+***	***-***	***+***	***Stable***
2a	+	-	+	-	+	+ / 0	+	Unstable
2b	*+*	*-*	*+*	*-*	*+*	*-*	*+*	*Stable*
3	*+*	*-*	*-*	*+*	*-*	*-*	*+*	*Stable*
4	***+***	***0***	***+***	***0***	***+***	***-***	***+***	***Stable***
5	-	+	+	-	-	+ / 0 / -	-	Unstable
6	-	-	+	+	-	+ / 0 / -	-	Unstable
7	-	-	-	-	+	+ / 0 / -	-	Unstable
8	-	0	+	0	-	-	-	Unstable

For the determinant of the Jacobian matrix to be non-negative and, therefore, for the system to show long-run stability, we need $D = s_R h u_n - \gamma - c_{WR} > 0$. If this condition holds, we can be certain to have a negative trace and thus long-run dynamic stability, if $F = \gamma - i > 0$, which also implies that $C = c_{WR} + \gamma - s_R i > 0$, as shown in case 1. Cases 2b, 3 and 4 also meet the stability requirement of a negative trace and a positive determinant of the Jacobian, even if the condition $F = \gamma - i > 0$ is not met. However, cases 2b and 3 are either implying a negative long-run equilibrium primary deficit-capital ratio (b^{**}), which means that governments would have to run primary surpluses, or a negative government debt-capital ratio (λ^{**}), which means that governments would have to hold net financial assets. Both cases are impossible in our simple model without taxation. Case 4 is a borderline long-run stable equilibrium, in which $F = \gamma - i = 0$ is associated with a zero primary government deficit- but a positive government debt-capital ratio in the long-run equilibrium.

For long-run stability we need, on the one hand, that the autonomous growth rate of government expenditures, which determines the long-run growth rate of the economy, does not fall short of the interest rate. This is the same as in the model by Hein (2018), which does not include a normal rate of capacity utilisation. On the other, this growth rate should not exceed a maximum given by the saving rate out of profits at the normal rate of utilisation minus the propensity to consume out of wealth. This means that for stability of economically meaningful values for the long-run equilibrium, it is required that:[7]

$$i \leq \gamma < s_R h u_n - c_{WR} \tag{36}$$

Without Harrodian instability, the inclusion of an interest rate on debt and consumption out of wealth provide more narrow constraints for the growth rate of autonomous expenditures to generate stable long-run equilibria than in the simple model presented in Section 2. Finally,

[7] Dutt (2020) has obtained an equivalent result for a model without consumption out wealth but with tax rates on wages and profits.

we can introduce Harrodian instability back into the model and replace equation (9) by the Harrodian equation (17):

$$\hat{\alpha} = \mu\beta(u^{*} - u_{n}), \quad \mu > 0 \tag{17}$$

The Jacobian of the three dimensional dynamics from equations (27), (17) and (28), taking into account the goods market equilibrium from equation (24) is given by:

$$J = \begin{bmatrix} \frac{\partial \hat{b}}{\partial b} & \frac{\partial \hat{b}}{\partial \alpha} & \frac{\partial \hat{b}}{\partial \lambda} \\ \frac{\partial \hat{\alpha}}{\partial b} & \frac{\partial \hat{\alpha}}{\partial \alpha} & \frac{\partial \hat{\alpha}}{\partial \lambda} \\ \frac{\partial \hat{\lambda}}{\partial b} & \frac{\partial \hat{\lambda}}{\partial \alpha} & \frac{\partial \hat{\lambda}}{\partial \lambda} \end{bmatrix} \tag{37a}$$

$$J^{**} = \begin{bmatrix} \frac{-\beta}{A} & \frac{-s_R h}{A} & \frac{-\beta B}{A} \\ \frac{\mu\beta}{A} & \frac{\mu\beta}{A} & \frac{\mu\beta B}{A} \\ \frac{AC - \beta D}{AD} & \frac{-s_R h}{A} & \frac{-ACF - \beta BD}{AD} \end{bmatrix} \tag{37b}$$

Again, we have to check the Routh-Hurwitz conditions for local stability of this three dimensional dynamic system, for which we need:

$$Det(J^{**}) = \frac{-\mu\beta C^2}{AD} < 0 \tag{38}$$

$$Tr(J^{**}) = \frac{-\beta D(1 - \mu + B) - AFC}{AD} < 0 \tag{39}$$

$$Det(J_1^{**}) + Det(J_2^{**}) + Det(J_3^{**}) = \frac{\beta\{C^2 + \mu[D(B+1) - CF]\}}{AD} > 0 \tag{40}$$

$$\begin{gathered} -Tr(J^{**})[Det(J_1^{**}) + Det(J_2^{**}) + Det(J_3^{**})] + Det(J^{**}) \\ = \frac{\beta\langle[\beta D(1 - \mu + B) + AFC]\{C^2 + \mu[D(B+1) - CF]\} - \mu AC^2 D\rangle}{A^2 D^2} > 0 \end{gathered} \tag{41}$$

As in the previous section, we will only discuss the long-run stability of economically meaningful, i.e. positive long-run equilibrium values for the government primary deficit- and the government debt-capital ratio in equations (31) and (32). Together with the stability condition for the goods market equilibrium this implies:

- $A = s_R h - \beta > 0$,
- $B = i(1 - s_R) + c_{WR} > 0$,
- $C = c_{WR} + \gamma - s_R i > 0$,
- $D = s_R h u_n - \gamma - c_{WR} > 0$,
- $F = \gamma - i > 0$.
- $F < C$, because $s_R < 1$ and $c_{WR} > 0$,
- $F + B = C$.

Therefore, for the R-H stability conditions we obtain the following results:

1. As can be seen in equation (38), the first stability requirement of a negative determinant is always met for positive long-run equilibrium values of the government primary deficit- and the government debt-capital ratio, because A, C and D are all positive.
2. The second condition of a negative trace is dependent upon μ, the Harrodian instability parameter, as can be seen in equation (39). For this condition to be met we need: $\mu < 1 + B + \frac{ACF}{\beta D}$, where the right hand side is large than one.
3. Condition three of a positive sum of the determinants of the three second-order principal sub matrices will be met according to equation (40), if $\mu > \frac{-C^2}{D(B+1)-CF}$. This is always true, if $D(B+1) - CF > 0$. If, on the other hand, $D(B+1) - CF < 0$, then we need $\mu < \frac{1}{1-\left(\frac{CB+D(B+1)}{C^2}\right)}$, where the right hand side is strictly positive and larger than one.
4. Demonstrating the conditions under which equation (41) is positive—thereby fulfilling the fourth and final R-H stability condition—is much more involved than the analysis hitherto. In particular, while it is easy in theory to derive the limits imposed upon μ by this fourth condition, the derived limits are difficult to compare with the limits implied by the second and third conditions. However, since a value of μ up to $1 + B$ satisfies the second and third conditions, we can simplify this problem by determining the conditions under which $\mu < 1 + B$ also satisfies this fourth condition and thus implies overall systemic stability. Given the length of the resulting mathematical treatment, we relegate it to the appendix and report the main results here. We find that if $D(B+1) - CF \leq 0$, then any value of $\mu < 1 + B$ will always satisfy this fourth condition. If, on the other hand, $D(B+1) - CF > 0$, then $\mu < 1 + B$ is possible, but with the caveat that the closer μ is to $1 + B$, the closer β must be to $s_R h$ (i.e. the closer *A* must be to zero) in order to fulfil this fourth R-H condition.

For the model with Harrodian instability, in order to obtain economically meaningful and stable long-run equilibrium values for the government primary deficit- and government debt-capital ratios, first, we need the same requirement as specified for the model without Harrodian instability, given in condition (36). The rate of interest provides a lower bound for the autonomous growth rate of government expenditures, which determines the long-run growth rate of the economy, and the saving rate out of profits at the normal rate of utilisation minus the propensity to consume out of wealth sets an upper bound. Second, however, a key difference now emerges, in that stability is more likely to be ensured as the growth rate of autonomous government expenditures approaches (but does not meet) this upper bound and the interest rate tends to zero. This follows because such a high value of γ and low value of i imply the case $D(B+1)-CF \leq 0$, which provides stability for any value of the Harrodian instability parameter, μ, up to $1+B$. An appropriate policy mix of a low interest rate and an adequately high growth rate of autonomous government expenditure would thus allow the Harrodian instability parameter to exceed one, up to $\mu < 1+B$, and thus contribute to overall stability.[8] In the alternative case $D(B+1)-CF > 0$ the system may still be stable, but this relies on parameters largely out of policymakers' control ($s_R h, \beta, \text{and } \mu$) taking on the stability-ensuring values by coincidence.

Thus, the main novel results from this model with consumption out of wealth, interest payments and Harrodian instability are as follows. First, we have shown that, under certain conditions, the system may exhibit local stability even for some values of the Harrodian instability parameter greater than one, extending the limit found in Lavoie (2016). Second, the policy instruments most directly under the control of policymakers have important implications for overall stability. For the greatest likelihood of overall stability, the interest rate need not only be lower than the growth rate of autonomous expenditures, but as low as possible. Likewise, the growth rate of autonomous expenditures should be as close as possible to its upper bound. Generally, then, the monetary policy implication of our findings is in line with those who advocate 'parking' the interest rate at a low and stable value (Rochon/Setterfield, 2007). Furthermore, proposals where fiscal policies follow an expenditure rule in order to promote growth at full employment levels (Hein 2018, Hein/Stockhammer 2010) may have the added benefit of increasing the stability of the system, even in the face of stronger Harrodian instability.

4. Conclusions

We have started from the observation that the Sraffian supermultiplier models, as well as Kaleckian distribution and growth models that incorporate autonomous demand growth in order to cope with Harrodian instability, have paid little attention to the financial side of such growth and to the issue of its stability. Therefore, our attempt has been to link the issue of Harrodian instability in Kaleckian models driven by non-capacity creating autonomous demand growth with the associated financial dynamics.

[8] Of course, it has to be conceded that, while a low interest rate allows the Harrodian instability parameter to exceed one, on the one hand, it also reduces the upper bound for this parameter of $1+B$, on the other hand, since $B = i(1-S_R)+c_{WR}$.

For this purpose we have used Lavoie's (2016) model as a starting point and have replaced autonomous consumption growth by government expenditure growth. In the first version with zero interest rates and no consumption out of wealth we have found that the inclusion of deficit and debt dynamics imposes an upper limit on the growth rate of autonomous government expenditures for stability purposes. This limit is given by the saving rate at normal capacity utilisation ($\gamma < s_\pi h u_n$). When Harrodian instability is introduced to this simple model, the same limit on the growth rate of autonomous government expenditures is found, and the upper bound on the Harrodian instability parameter is one, $\mu < 1$, both as in Lavoie (2016).

In the second step, we have then introduced interest payments on government debt as well as consumption out of real and financial wealth into the model. Switching off Harrodian instability in the first version, by assuming that firms 'rationally expect' the long-run growth rate of the system given by autonomous government expenditure, has shown that the model may generate stable long-run equilibria, if two conditions are met. First, the autonomous growth rate of government expenditures determining the long-run growth rate of the economy should not fall short of the exogenous monetary interest rate. Second, this growth rate should not exceed a maximum given by the saving rate out of profits in long-run equilibrium minus the propensity to consume out of wealth. These conditions were also needed when we re-introduced Harrodian instability into the model, but with the added caveat that stability is more likely to be ensured the lower the interest rate is and the closer the growth rate of autonomous government expenditures is to its upper bound. Furthermore, for long-run equilibria to be stable, the maximum value for the Harrodian stability parameter of $1 + i(1 - s_R) + c_{WR}$, which is higher than in Lavoie's model, should not be exceeded.

Summing up, our exercises in an extremely simple model framework have shown that the financial dynamics, necessarily associated with autonomous demand growth, have an important impact on long-run stability and sustainability of growth driven by autonomous demand. Empirically, and given the experiences from the 2007-9 financial and economic crises, as well as from the following Eurozone crisis, this should not be surprising. We have provided a starting point for the integration of these concerns into simple analytical distribution and growth models driven by autonomous demand. Further analysis may build on this.

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Appendix: Conditions imposed by the fourth Routh-Hurwitz stability criterion for the second model in Section 3

In this appendix, we aim to show when the fourth Routh-Hurwitz condition (R-H4) holds. We begin by reprinting R-H4 in full:

$$-Tr(J^{**})[Det(J_1^{**}) + Det(J_2^{**}) + Det(J_3^{**})] + Det(J^{**})$$
$$= \frac{\beta\langle[\beta D(1-\mu+B) + AFC]\{C^2 + \mu[D(B+1) - CF]\} - \mu AC^2 D\rangle}{A^2D^2} > 0 \tag{41}$$

We define the function $Z(\mu)$, which is the component of equation (41) that determines the sign of the left hand side, as follows:

$$\begin{aligned} Z(\mu) = &- \beta D[D(B+1) - CF]\,\mu^2 \\ &+ \{[\beta D(1+B) + AFC][D(B+1) - CF] - s_R h C^2 D\}\,\mu \\ &+ C^2[\beta D(1+B) + AFC] \end{aligned} \tag{A1}$$

Finding the roots of this quadratic equation by the usual means does not simplify neatly such that we can compare the range(s) of μ that satisfy R-H4 with ranges of μ that satisfy the second and third R-H conditions derived in Section 3. We therefore undertake the following analysis.

Let us define *P* and *Q*, such that $P \equiv D(B+1) - CF$ and $Q \equiv \beta D(1+B) + AFC$. Note that P may be positive or negative, whereas Q is always positive and increases as P increases, since $Q = \beta P + s_R hCF$. With these variables so defined, we arrive at:

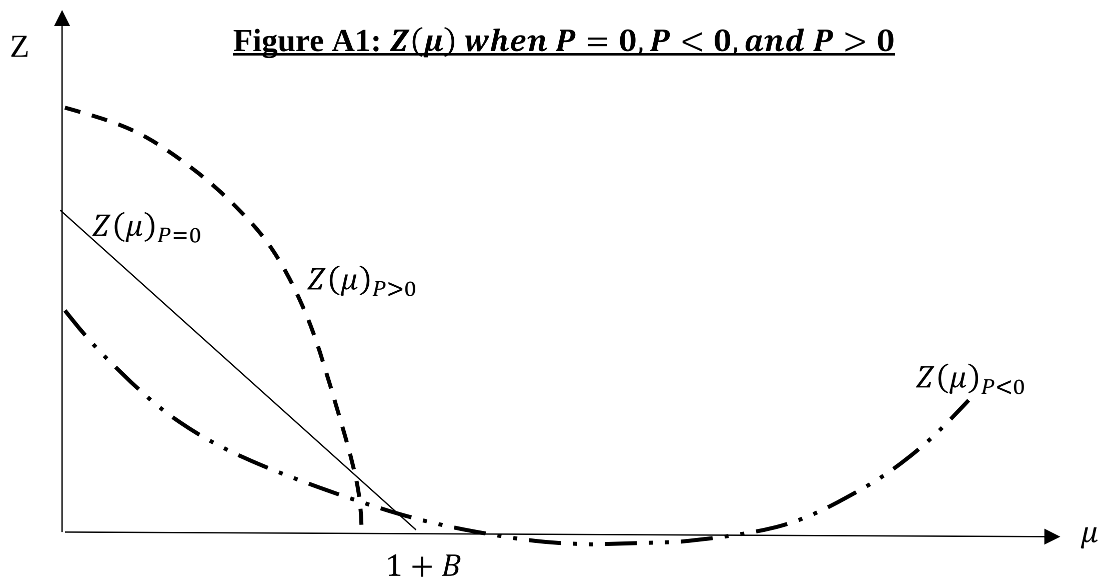
$$Z(\mu) = -\,\beta DP\,\mu^2 + (PQ - s_R hC^2 D)\,\mu + C^2 Q \tag{A2}$$

From inspection of equation (A2) it is clear that

- $Z(\mu)$ is linear in μ in the special case where $P = 0$.
- $Z(\mu)$ is quadratic in μ in general and the resulting parabola is concave down if $P > 0$ and convex up if $P < 0$.
- $Z(0) > 0$, implying the function always has a positive y-intercept.

Given the pivotal role *P* plays in determining $Z(\mu)$, it will serve us well to consider three cases, as displayed in Figure A1: The first where $P = 0$, the second when $P < 0$, and the third where $P > 0$. Considering each case, we can find the values of μ for which $Z > 0$ and so RH4 holds.

Figure A1: $Z(\mu)$ when $P = 0, P < 0, and\ P > 0$



Case 1: *P = 0*

In the special case where *P = 0*, $Z(\mu)$ is linear and given by:

$$Z(\mu) = -s_R hC^2 D\, \mu \ + s_R hC^3 F \tag{A3}$$

From which it follows that $Z(\mu) = 0$ when

$$\mu = \frac{CF}{D} \tag{A4a}$$

However, since *P = 0* and thus $D(B + 1) = CF$, we know equation (A4a) is equivalent to

$$\mu = 1 + B \tag{A4b}$$

Hence, when *P = 0*, $Z(\mu) > 0$ and RH4 holds when $0 < \mu < 1 + B$.

Case 2: P < 0

When *P < 0*, we know that $Z(\mu)$ is convex up and that the minimum of *Z* occurs when $\mu > 0$, since the coefficient on the linear term in equation (A2) is unambiguously negative, and the coefficient on the quadratic term is positive. We can show that $Z(\mu)_{P<0} > 0$ for values of μ in the range of $0 < \mu \leq 1 + B$ in two steps. First, we will show that $Z(1 + B)_{P<0} > 0$. Equation (A5) shows that the value of *Z* at $1 + B$ is

$$Z(1 + B) = AFC\{C^2 + (B + 1)P\} - (B + 1)AC^2D \quad \text{(A5)}$$

This neatly simplifies to

$$Z(1 + B) = ACP[B(F - 1)] \quad \text{(A6)}$$

Since $P < 0$ and $F < 1$ for any realistic values of *F*, it follows that $Z(1 + B) > 0$.

The second step consists of demonstrating that the first derivative of $Z(\mu)_{P<0}$ at $1 + B$ is negative, i.e. $Z'(1 + B)_{P<0} < 0$. Note that this step is necessary as it rules out the possibility that the minimum of $Z(\mu)_{P<0}$ occurs before $1 + B$ and, hence, shows that $Z(\mu)_{P<0}$ must be positive for all values of μ between 0 and $1 + B$. Thus, we take the first derivative of *Z,* evaluate it at $1 + B$ and check when it is less than zero.

$$Z'(1 + B) = -\,2\beta DP\,(1 + B) + \,P[\beta P + s_R hCF] - s_R hC^2D < 0 \quad \text{(A7)}$$

This can be reduced to

$$\beta < s_R h\,\frac{C^2D + F[CF - D(B + 1)]}{C^2F^2 - D^2(B + 1)^2} \quad \text{(A8)}$$

Condition (A8) is always fulfilled because $\beta < s_R h$ from the goods market equilibrium stability condition and the term multiplying $s_R h$ is positive (since *P<0* and thus *CF > D(B+1)*) and, it can be shown, greater than one. Hence, $Z'(1 + B)$ is always negative when *P < 0*.

From this demonstration it follows that when *P < 0*, RH4 will be fulfilled for any value of $\mu \leq 1 + B$. Note there may also be values of μ greater than $1 + B$ that fulfil RH4. However, these are less relevant for our purposes as they will likely conflict with the restrictions on μ given by the R-H conditions 2 and 3 in Section 3.

Case 3: P > 0

When *P > 0*, we know that $Z(\mu)$ is concave down and that the maximum of Z could occur when μ is negative or positive, as it depends on the sign of the coefficient on the linear term in equation (A2). From equation (A6) it is clear that if $\mu = 1 + B$, then $Z(\mu)_{P>0} < 0$ and R-H4 is not satisfied. Hence, the upper limit to μ when *P > 0* must be less than 1+B.

We can show that the maximum value that μ may take without violating RH4 is dependent on the size of *A* (i.e. the size of β relative to $s_R h$).[9] If β is very close to $s_R h$, and hence *A* is close to zero, then the upper bound on μ will be close to $1 + B$:

[9] Note that β does not enter into our definitions of B, C, D, F or P and so analysing these limits is relatively straightforward.

$$\lim_{\beta \to s_R h} Z(\mu)_{P>0} = s_R h D\{ -P\mu^2 + [P(1+B) - C^2]\mu + C^2(1+B)\} \tag{A9}$$

The positive root of the concave quadratic in equation (A9) is $1+B$. Hence, the closer β is to $s_R h$, the closer the upper bound of the R-H4-satisfying value of μ will be to $1+B$.

On the other hand, if β approaches zero, then $Z(\mu)_{P>0}$ is increasingly well approximated by the line given by

$$\lim_{\beta \to 0} Z(\mu)_{P>0} = s_R h C\ [\ (PF - CD)\mu + C^2 F] \tag{A10}$$

Equation (A10) is positive for $\mu < \frac{C^2 F}{CD - PF}$. Clearly, then, if β and *F* are very small, the largest value of μ that satisfies R-H4 must also be small.

Generally, we can show that for any value of μ between 0 and $1+B$, $Z(\mu)_{P>0}$ will fall as β falls. This is tantamount to showing that the derivative of $Z(\mu)_{P>0}$ with respect to β

$$\frac{\partial \mathrm{Z}}{\partial \beta} = -DP\mu^2 + P^2\mu + C^2 P \tag{A11}$$

is positive between 0 and $1+B$. Clearly, equation (A11) is positive when $\mu = 0$. Hence, by showing that it is also positive when $\mu = 1+B$, it must follow that $\frac{\partial \mathrm{Z}}{\partial \beta} > 0$ for every value of μ in between 0 and $1+B$ as well, since the quadratic in equation (A11) is concave down. Inserting $\mu = 1+B$ into equation (A11) and simplifying yields:

$$\left.\frac{\partial \mathrm{Z}}{\partial \beta}\right|_{1+B} = PB[D(1+B) + C(1-F)] > 0 \tag{A12}$$

In sum, when *P > 0*, any value of μ, such that $0 < \mu < 1+B$, may satisfy R-H4 for a sufficiently high value of β (or, equivalently, a sufficiently low value of *A*).[10]

[10] For example, we can show that R-H4 is satisfied for $\mu \leq 1$ if β is large enough such that $\frac{\beta}{s_R h} > \frac{C(P-CD)}{P(D+C)}$, where the term on the right hand side is strictly lower than one.