Testing fundamentalist-momentum trader financial cycles. An empirical analysis via the Kalman filter

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Abstract

This paper proposes an empirical test for Minskyan financial cycles in asset prices, driven by the interaction of fundamentalist and momentum traders. Both price strategies are unobserved and can be modelled in a state space model. We use the Kalman filter to identify the two pricing strategies and evaluate whether the conditions for the existence of cycles hold. The model is estimated for four major OECD countries, the UK, France, Germany and the USA, for equity and housing prices for the period 1970-2017 using annual data. We find evidence of cycles in the equity market for all four countries and for housing prices, in the UK, France and the USA but not in Germany. Our results provide empirical support for the existence of endogenous financial cycles on asset markets.

Keywords: Financial cycles, Minsky, Momentum traders, Kalman filter.

JEL codes: C32, E32, G40.

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1 Introduction

During the Great Moderation period, standard macroeconomic models paid limited
attention to financial cycles. Borio (2014) criticizes the fact that the New Keynesian
dynamic stochastic general equilibrium paradigm has regarded finance as a veil that it has
been ignored in the studies of business fluctuations and that financial crises were
interpreted as the result of exogenous shocks. By contrast, since the global financial crisis,
financial factors feature prominently (Nikolaidi, 2014; Mian et al., 2017; Jordá et al., 2016;
Stockhammer et al., 2019a; Kohler, 2019). This approach builds on Minsky’s financial
instability hypothesis (Minsky, 1985) and behavioral finance (Shiller, 2003), which regard
financial cycles and market inefficiency as the outcome of endogenous forces.

Minsky emphasizes the role of financial factors in a capitalist economy, characterized
by the gradual emergence of endogenous financial fragility which eventually turns the
boom into a bust (Ferri and Minsky, 1992; Vercelli, 2000). Nikolaidi and Stockhammer
(2017), in a recent survey of Minskyian theory, identify two families of Minsky models. In
the first, the dynamics emerge from the interaction of financial factors (usually debt or
interest rate) and a real variable (typically investment). A second family describes cycles
as the outcome of the interaction to two asset pricing strategies on financial markets. This
latter family overlaps with behavioral economics models (Franke and Westerhoff, 2017).

The existing empirical literature on the financial instability hypothesis is sparse and
focuses on the first family. Schroeder (2009), Mulligan (2013), Nishi (2016), and Davis et
al. (2017) seek to identify the hedge, speculative and Ponzi states of a firm’s condition for
different countries and economic sectors. Other studies have explored the impact of debt
on aggregate demand (Palley, 1994; Kim, 2013, 2016). Stockhammer et al. (2019a; 2019b)
formally test whether financial-real interactions give rise to endogenous cycles. As
financial variables, they consider the interest rate as well as business and household debt.
However, there are no empirical Minsky studies that incorporate an active role for asset
prices with the crucial role of the evaluation strategies of the agents. This paper will deal
with the second group, the momentum trader models.

Momentum trader models suggest that there is heterogeneity in the expectation
formation on financial markets. These can be grouped into fundamentalist (mean
reverting) and momentum trader (also: extrapolative traders) pricing strategies. Under
certain conditions (Beja and Goldman, 1980) the interaction between the two will generate
cycles in asset prices. This argument is in line with behavioral economics whereby changes
in price occur not for fundamental reasons but because of heuristics. This theory
emphasizes psychological elements in the decisions of traders such that price booms
rooted in feedback mechanisms rather than changes in fundamentals can arise (Schleifer
and Summers, 1990; Shiller, 2003; Vikash et al., 2015).

Importantly, these pricing strategies, by their nature, cannot be directly observed but
they will cause a response in the observed data. The contribution of this paper is to provide
an empirical test for endogenous financial cycles that emerge from the interaction of the
two latent pricing strategies. To achieve this, we use the Kalman filtering in a state space
model with the aim of explaining the dynamics of asset prices in a context of an
unobserved component model. Kalman-filtering is a recursive dynamic procedure used
to estimate time dependent structural parameters of linear systems. It is used routinely in

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1 The indebtedness of firms is expressed in Minsky’s categorization of firms as the hedge,
speculative and Ponzi ones. Based on the relationship between cash flow and debt service
requirements, firms gradually shift from hedge to speculative and Ponzi regimes, thereby
generating over-indebtedness and higher financial fragility.
economics to estimate output gaps and the NAIRU (Boone, 2000; Rusticelli, 2014) and to decompose the trend and cyclical components of the GDP and other economic time series (De Winter et al., 2017; Klinger and Weber, 2019). We estimate the parameters associated with the two price strategies to analyze the presence of financial cycles and the relative shares of the two economic agents in the market. This method serves as a valuable instrument to search for empirical evidence of endogenous financial cycles in a context of an unobserved components model. A precondition for using the Kalman filter is that the model is linear. This is a shortcoming as some momentum trader models are non-linear (Ryoo, 2010; Westerhoff, 2006a), in particular the share of fundamentalist and momentum traders may be endogenous (Hommes, 2006; Franke, 2008; De Grauwe, 2008; 2012). Our model should be interpreted as a linear approximation.

The model is estimated for the UK, France, Germany and the USA using the times series of equity and house prices over the period 1970-2017. We analyze equity prices both because they play a key role in Minsky models and because they are frequently used as asset price indicators for macroeconomic analysis. The choice of housing prices is due to the increasing interest in real estate prices in the Minskyan framework since the global financial crisis (Ryoo, 2016). Our results provide evidence of financial fluctuations in the equity market for the UK, France, Germany and the USA, with the highest price overshooting in economies with market-based financial systems, respectively the UK and the USA. Regarding the house prices, we find robust evidence of cyclical fluctuations in the UK, France and the USA with the highest price overshooting in the USA. For Germany, we find evidence for cycles, but the estimates are not statistically significant.

The paper is organized as follows. Section 2 reviews both the relevant theoretical and empirical literature. Section 3 presents the model and clarifies the conditions under which oscillations arise. Section 4 presents data and our econometric approach. Section 5 discusses the estimation results. Section 6 concludes with final considerations and directions for future research.

2 Review of the relevant literature

Since the 1980s, followers of the post-Keynesian school of economics have developed the economic ideas of Hyman Minsky in formal mathematical models. However, despite the great number of theoretical studies (see for example Taylor and O’Connell, 1985; Vercelli, 2000; Foley; 2003; Charles, 2008, Ryoo, 2010; 2013; Köhler, 2019 among others), there are few empirical studies on the financial instability hypothesis. Section 2.1 revisits the theoretical and empirical papers on Minsky’s theory. In section 2.2 we review the behavioral theory which highlight the role of heuristic strategies that can give rise to instability and fluctuations and the empirical literature on heterogeneous agents models.

2.1 Contributions on Minsky’s theory

Due to the lack of agreement on the formal presentation of Minsky’s argument, the financial instability hypothesis has been formalized and interpreted in different ways. Minsky models can be grouped into debt cycles and asset price cycles. In the first, the dynamics of debt or interest rate is central in the analysis with no role assigned to asset prices (see for example Charles, 2008, Fazzari et al., 2008 and Nikolaidei, 2014). In the second group, asset prices play the key role (see for example, Taylor and O’Connell 1985 and Ryoo 2016). In the standard version of the debt cycles, the model consists of a pro-cyclical debt ratio and a long-term negative effect of debt on investment which interact
to generate cycles (Stockhammer, 2019). This idea is developed using diverse mechanisms and theoretical foundations: we can list i) the Kalecki-Minsky models; ii) Kaldor-Minsky models; iii) Goodwin-Minsky models; iv) credit rationing models; v) endogenous target debt ratio models; and vi) Minsky-Veblen models. In asset prices cycles we can distinguish between the equity price Minsky models (Taylor and O’Connor (1985); Ryoo, 2010, 2013) and the real estate price Minsky models (Ryoo, 2016). Within this group, asset cycles are characterized by the speculative activity of agents based on expected capital gains that lead to an unsustainable bullish period which ultimately turns into a bust. In this class of models, two valuation strategies interact, sometimes referred to as fundamentalist and momentum traders, with momentum traders providing the overshooting force. The interaction between the stabilizing of fundamentalists and the destabilizing of chartists speculators generates oscillation dynamics (Chiarella and Di Guilmi, 2011; Ryoo 2010, 2013; Sordi and Vercelli, 2012).

A small but growing body of literature has empirically examined the impact of financial variables on aggregate demand or their ability to cause crises. Palley (1994) and Kim (2013; 2016) estimate vector autoregressive (VAR) and vector error correction (VEC) models with GDP and household debt and report positive short-run feedback effects and negative long-run feedback effects of household debt on output. Greenwood-Nimmo and Tarassow (2016), with a policy-oriented Minsky model, examine the implications of monetary and macro-prudential shocks for aggregate financial fragility using a sign restricted VAR model.

The existing studies all focus on the interaction of the goods market and financial markets as the source of instability or cyclical phenomena. Moreover, Palley (1994), Kim (2013; 2016) and Greenwood-Nimmo and Tarassow (2016) do not test explicitly for endogenous cycles. Only recently, Stockhammer et al. (2019a) explicitly test the real-financial interaction mechanism and evaluate whether it gives rise to endogenous cycles. They start from a reduced form system of simultaneous equations in which a real variable and a financial variable interact with each other. Two conditions guarantee endogenous oscillations in a debt-burdened growth: complex eigenvalues and negative sign of the coefficient’s product of the Jacobian matrix. This means that from the interaction between the two state variables of the system an increase in one variable (the real one) induces an acceleration of the second variable (the financial one) which in turn drags down the first. They find evidence for financial-real interactions at high frequencies between GDP and interest rate and a low frequency between GDP and business debt. No evidence between GDP and household debt interaction is found. In the same vein, Stockhammer et al. (2019b), with historical macroeconomic data, estimate a Vector Autoregressive Moving Average model (VARMA), to investigate whether business cycles are driven by corporate debt or by mortgage debt. They find that the USA economy has experienced corporate debt-driven Minsky cycles over the sample period. For the UK the leverage ratio is procyclical, but no robust evidence for debt- burdened growth is found. Again, the estimation using mortgage debt yields no evidence for mortgage debt-driven Minsky cycles.

In summary, all the empirical works discussed above explore the empirics of Minskyan financial fragility but none of these studies account for the fundamental role played by asset prices. In order to fill this gap, we intend to empirically examine weather the asset prices dynamics in a context of different valuation price strategy is the driver of cyclical behavior. At the same time, the idea that speculative behavior can have a direct effect on asset price dynamics is in line with behavioral theory. In this sense, our work can support the increasing number of theoretical studies that are based on behavioral arguments.

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2 The literature is too vast to be cited in extenso. See Nikolaidi and Stockhammer (2017) for this type of classification and references.
2.2 Speculative behavior: a brief review of the theoretical and empirical models

Theoretical studies in which the speculative thinking among investors plays a fundamental role in the determination of asset prices have an historical background in economics. Beja and Goldman (1980) in their seminal work present a dynamic model of the asset prices process in a disequilibrium setting. They distinguish between fundamentalist and speculative traders who act on their perception of the current price trend, i.e. they take into account information (past prices) which is unrelated to economic fundamentals. The speculation on the asset price-trend generates endogenous instabilities and oscillations in the price. Beja and Goldman (1980) thus prepares the ground for behavioral theory (Schleifer and Summers, 1990; Shiller, 2003; Vikash et al., 2015) and a variety of heterogeneous agents models (see e.g. Hommes, 2006 and Franke, 2008 for an overview).

After the global financial crisis, the behavioral argument has received increasing attention and some of its insights have been incorporated in macroeconomic models. These theoretical studies range from the Behavioral New-Keynesian Models (BNKM) (De Grauwe 2008, 2012; Bofinger et al., 2013) to the linear and non-linear dynamic models of speculative market in a disequilibrium setting (Westerhoff, 2006a; 2006b; Lines and Westerhoff, 2006; Dieci and Westerhoof, 2012).\(^3\) Despite the different paradigms, all these works allow for heterogeneity among agents. With regard to the BNKM, De Grauwe (2008; 2012) and Bofinger et al. (2013) highlight the role of heuristics in real and financial market. The agents may use fundamentalist or extrapolative rules to form their expectations. Fundamentalists act on the basis of fundamental information and process information rationally. In contrast, extrapolators base their expectations on past dynamics. They show by means of numerical simulation how the extrapolative formation rules of agents produce waves of optimism and pessimism in an endogenous way thus providing an explanation of the observed oscillation. In contrast to the paper by Beja and Goldman (1980), these authors introduce a time-variant selection mechanism à la Brock and Hommes (1998), thanks to which agents evaluate the performance of the rule to perform. Parallel to these, Westerhoff (2006a; 2006b; 2008) and Lines and Westerhoff (2006) present more general disequilibrium dynamic models. Building on the multiplier-accelerator models of Samuelson (1939) they show how economic activity endogenously depends on extrapolative and mean-reverting behavior, thus emphasizing the role of heuristics in the generation of the business cycle. Dieci and Westerhooff (2012) analyze the house price dynamics in a nonlinear speculative discrete time dynamic model. Total demand for housing is created as an interaction between real and speculative demand, where the real demand decreases in price while the speculative demand is driven by price dynamics and depends on extrapolative and mean-reverting speculative strategies.

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\(^3\) The non-rational behavior is formalized assuming different behavioral biases. In De Grauwe (2008; 2010) momentum traders extrapolate variable of interest from the past into the future considering observed past values. The same in Westerhoff (2008) with different autoregressive process. In Westerhoff (2006a; 2006b) and Lines and Westerhoff (2006) extrapolators base their beliefs on the observed past period and fundamental value.
In contrast to the considerable number of theoretical studies, the empirical literature is rather limited and there is no consensus on the estimation methodology. Franke and Westerhoff (2017) note two approaches: direct and indirect. The first method employs an economic survey to measure the sentiments of a specific group of the population, typically the momentum traders, and thus explain their behavior. The second considers a model as a whole and strives to estimate all its parameters in one effort. With reference to the latter we can distinguish between two types of inference method. In the first, key structural features of agent-based models can be estimated in a straight way. Depending on the complexity of the models, we can list the nonlinear least squares, the maximum and quasi-maximum likelihood among others (Kulacka and Barunick, 2017; Chiarella et al., 2014; Westerhoff and Reitz, 2003); in line with the work of Frankel and Froot (1990) in our work the fraction of the two types is fixed in time. With reference to the second method, estimation based on simulating artificial data from the model is used instead. The most frequently estimation method used is the method of simulated moments (MSM), (Franke and Westerhoff, 2011; Franke and Westerhoff, 2012). Estimation by MSM means searching for the parameter values of a model that minimize the distance between the simulated and the empirical counterparts. Through simulation runs it is possible to depict phenomena which are consequence of behavioral biases, such as volatility clustering, long memory effects, and a herding behavioral predisposition.

Empirical works of this type have been applied to different markets, such as equities, housing and foreign exchange market. Chiarella et al. (2014), Lof (2012; 2015) and Hommes and Veld (2015) suggest that heuristics perform very well in describing the dynamics of the stock market prices. Westerhoff and Reitz (2003) and De Jong et al. (2010) analyze the exchange rates market. In general, these works suggest that sentiment dynamics are important in explaining stylized facts observed in financial time series and in replicating observed anomalies in financial markets.

Along this line of research, our paper highlights the heterogeneity among agents and seeks to empirically identify the different evaluation price strategies. The behavioral models mentioned above do not provide evidence of cycles emerging directly from the data as a consequence of behavioral heuristics. The present paper proposes an estimation methodology for the empirical validation of endogenous cycles which has not yet been explored in the literature. We consider the beliefs of the agents as unobserved state components from which, through a state space model formulation, the endogeneity of fundamentalist-momentum trader cycles can be directly evaluated from the data. To achieve this, we use the maximum likelihood estimation (MLE). Unlike the indirect simulated-based estimation, as for the MSM, with MLE direct analytically estimation techniques are feasible. However, differently from previous studies, we work in a state space model. Numerical techniques trough the Kalman filter algorithm are applied so that the maximized value of the log likelihood function can be reached and parameters can be recovered. Besides the tractability of the model, the main advantage of this framework is that, filtering information on unobserved states, it is able to test whether behavioral rules lead to the cyclical dynamics in the observed asset prices.
3 The model

In this section we present the model and describe the proposed modelling strategy. Let the observed asset price be $P_t$, for equity asset and housing price, dependent on the weighted sum of two unobserved stochastic dynamic components, respectively the fundamental price strategy $P_t^f$ and the momentum price strategy $P_t^m$:

$$P_t = \gamma P_t^f + (1 - \gamma)P_t^m \quad 0 \leq \gamma \leq 1 \quad (1)$$

where the weights $\gamma$ and $1 - \gamma$ are the proportions of fundamentalists and extrapolative agents in the housing and equity market.

Regarding the fundamentalists, following the efficient market hypothesis, fundamental prices strategy cannot be based on past prices information and consequently, historical prices are of no value. In this sense, the fundamental price strategy is based on the fundamental price which is known by fundamentalists and updated with time. The fundamental value is intrinsic to the asset and based on the expected income stream. So the fundamentalist strategy can be defined in the following way:

$$P_t^f = P_{t-1}^f + \epsilon_t \quad \epsilon_t \sim N(0, \sigma_{\epsilon}^2) \quad (2)$$

where $\epsilon_t$ is the individual disturbance term which is normally distributed with mean zero and variance $\sigma_{\epsilon}^2$. Eq. (2) depicts a random walk. We assume that the fundamentalists believe that the price follows a random walk pattern. One implication of this is that, in the event of asset price boom, fundamentalists expect a return of actual prices towards the fundamental price (mean reversion).\(^5\) As to the momentum traders, we define their price in the following way:

$$P_t^m = P_{t-1} + \beta (P_{t-1} - P_{t-2}) + \eta_t \quad \beta \geq 0, \quad \eta_t \sim N(0, \sigma_{\eta}^2) \quad (3)$$

where $\beta$ denotes the actual extrapolation parameter which captures the agent’s price overshooting and $\eta_t$ is the individual disturbance term which is normally distributed with mean zero and variance $\sigma_{\eta}^2$. From Eq. (3), when the asset price is above (below) its value at previous time, it follows that the economic agent optimistically (pessimistically) believes in a further price increase (decrease). This form of price strategy can be defined as a form of speculation on the current price trend based on the extrapolation of past prices rather than by fundamental news. Given Eq. (1), the extrapolative price strategy can be rewritten in the following way:

$$P_t^m = \gamma (1 + \beta)P_{t-1}^f + (1 - \gamma)(1 + \beta)P_{t-1}^m - \gamma \beta P_{t-2}^f - \beta (1 - \gamma)P_{t-2}^m + \eta_t \quad (4)$$

We set

\[^4\] Momentum traders and extrapolative traders are used synonymously.
\[^5\] See Appendix A.
\[ a_{21} = \gamma (1 + \beta) \]
\[ a_{22} = (1 - \gamma) (1 + \beta) \]
\[ a_{23} = -\gamma \beta \]
\[ a_{24} = -\beta (1 - \gamma) \]

such that

\[ p_t^m = a_{21} p_{t-1}^f + a_{22} p_{t-1}^m + a_{23} p_{t-2}^f + a_{24} p_{t-2}^m + \eta_t \]

In the context of the unobserved component model, fundamental and extrapolative price strategies are unobserved state variables that have to be specified in a state space form. With this modelling strategy, we can reveal the nature and the cause of the dynamic movement of observed variables in an effective way. In fact, with a state space model it is possible to explain the behavior of an observed variable by examining the internal dynamic properties of the unobserved components. An essential feature of any state space model is that the state equation must be a first-order stochastic difference equation (Enders, 2016). In our model the observation equation of the state space model is

\[ P_t = (\gamma \quad 1 - \gamma \quad 0 \quad 0) \begin{pmatrix} p_t^f \\ p_t^m \\ p_{t-1}^f \\ p_{t-1}^m \end{pmatrix} \quad (6) \]

Taking into account Eq. (5) and Eq. (2) with \( a_{11} = 1 \), we have the transition equation of the state space model

\[
\begin{pmatrix}
  p_t^f \\
p_t^m \\
p_{t-1}^f \\
p_{t-1}^m
\end{pmatrix} = \begin{pmatrix}
a_{11} & 0 & 0 & 0 \\
a_{21} & a_{22} & a_{23} & a_{24} \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
p_{t-1}^f \\
p_{t-1}^m \\
p_{t-2}^f \\
p_{t-2}^m
\end{pmatrix} + \begin{pmatrix}
\epsilon_t \\
\eta_t
\end{pmatrix} \quad (7)
\]

In a compact form, we define

\[ P_t = H Z_t \quad (8) \]

\[ Z_t = A Z_{t-1} + \delta_t \quad \delta_t \sim N(0, Q) \quad (9) \]

where \( P_t \) is the observable asset price,
\[
Z_t = \begin{pmatrix}
p_f^t \\
p_t^f \\
p_t^m \\
p_{t-1}^f \\
p_{t-1}^m
\end{pmatrix}
\]

is the state vector,

\[
H = \begin{pmatrix}
\gamma & 1 - \gamma & 0 & 0 \\
\end{pmatrix}
\]

is the measurement matrix,

\[
A = \begin{pmatrix}
a_{11} & 0 & 0 & 0 \\
a_{21} & a_{22} & a_{23} & a_{24} \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\]

is the transition matrix and \( \delta_t \) is the vector containing the state disturbance of unobserved components, normally distributed with mean zero and variances collected in the diagonal matrix \( Q \).

The dynamic of the system is given by the transition equation which describes the evolution of the vector of unknown latent variables. Eigenvalues analysis can be performed to study the conditions for oscillations in our two-dimension discrete dynamic system associated with the two unobserved price strategies.\(^6\) We obtain the associated characteristic equation considering the following determinant of the transition matrix:

\[
\begin{vmatrix}
a_{11} - \lambda & 0 & 0 & 0 \\
a_{21} & a_{22} - \lambda & a_{23} & a_{24} \\
1 & 0 & -\lambda & 0 \\
0 & 1 & 0 & -\lambda
\end{vmatrix} = 0
\]

First of all, we have the following two eigenvalues

\[
\lambda_4 = a_{11} = 1 \in \mathbb{R} \quad \lambda_3 = 0
\]

In addition, regarding the other two eigenvalues, they must satisfy

\[
\begin{vmatrix}
a_{22} - \lambda & a_{24} \\
1 & -\lambda
\end{vmatrix} = \lambda^2 - a_{22} \lambda - a_{24} = 0
\]

from which

\[
\lambda_{1,2} = \frac{a_{22} \pm \sqrt{a_{22}^2 + 4a_{24}}}{2}
\]

\(^6\) See Appendix B.
In order to have an oscillating behavior, these two last eigenvalues have to be complex, so that we require

\[ \Delta = a_{22}^2 + 4a_{24} < 0 \]

i.e.:

\[ a_{24} < -\frac{a_{22}^2}{4} \quad (10) \]

When this is the case:

\[ \lambda_{1,2} = \frac{a_{22}}{2} \pm i \sqrt{-\frac{(a_{22}^2 + 4a_{24})}{2}} = a + ib \]

where \( i \) is the imaginary unit and \( a \) and \( b \) are real numbers. \( a \) is called the real part of the complex number and \( ib \) is the imaginary part. The complex number in the Cartesian form \( a + ib \) can be written written in the equivalent trigonometric \( \rho(\cos \omega \pm i \sin \omega) \). The positive number \( \rho = (a^2 + b^2)^{\frac{1}{2}} \) is called the modulus of the complex number (Gandolfo, 2009).

In order to have oscillations of constant amplitude we require

\[ \rho = 1 \]

i.e.:

\[ \sqrt{\left(\frac{a_{22}}{2}\right)^2 + \frac{-(a_{22}^2 + 4a_{24})}{4}} = 1 \]

from which

\[ a_{24} = -1 \]

Inserting in Eq. (10)

\[ -2 < a_{22} < 2 \]

Then, the conditions to have oscillating behavior of constant amplitude are

\[ a_{24} = -1 \]
\[ -2 < a_{22} < 2 \]

If the condition in Eq. (10) is respected, with \(-1 < a_{24} < 0\) (length of eigenvalues < 1) we have damped oscillations. With \(a_{24} < -1\) (length of eigenvalues > 1) we have explosive oscillations. Summarizing we have an oscillating system if

\[
|a_{11}| \leq 1 \quad \forall a_{21}, \forall a_{23} \quad a_{24} < -\frac{a_{22}^2}{4}
\]  

(11)

4 Data and econometric approach

The dataset, with annual frequency, consists of four OECD countries: the UK, France, Germany and the USA. We consider the time series of equity prices and housing prices with a sample size ranging from 1970 to 2017. For all the four countries, the source for equity and housing time series is the OECD database. We use deflated series for all the variables. Housing prices and equity prices series are deflated by the GDP deflator, which is taken from the Federal Reserve Economic Database for all the countries.7

In our model the driving forces behind the evolution of economic variables are not observable. In fact, asset price dynamics depend on different price strategies of economic agents. In a context of the unobserved components model, the estimation problem can be solved with the Kalman filter approach in a state space model formulation. The state space model and the Kalman filter go hand-in-hand: to use the Kalman filter, it is necessary to be able to write the model in the state space form. Once the model is in state space form, the recursive Kalman filter algorithm is used in calculating the optimal estimator of the state variables and in estimating the model parameters. Precisely, the parameters of the model are estimated by maximum likelihood using the prediction error decomposition approach where the one-step prediction and updating equations are calculated in a state space form using the Kalman filtering.8 Given the vector prediction errors and the variance-covariance matrix of the system, the log likelihood can be maximized.9 In other words, the Kalman filter makes possible to construct the likelihood function associated with a state space model to estimate the parameters of unobservable variables. In our case, this econometric methodology seems to be the most appropriate for its statistical characterization. In fact, it aims to model latent factors (price strategies) that cannot be measured directly but lead to the responses in observed data (asset prices).

In the econometric analysis we set \(a_{11}=1\) for the fundamentalists. For the momentum traders, the coefficients \(a_{21}, a_{22}, a_{23}\) and \(a_{24}\) are estimated. To obtain oscillations, conditions in Eq. (11) have to be respected. Moreover, we estimate \(\gamma\) to obtain the proportion of fundamentalists and momentum agents both in equity and housing market. Once we obtain our estimation results, with \(a_{22}, a_{23}\) and \(\gamma\) it is possible to obtain \(\beta\) using Eq. (5).

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7 For the econometric analysis all the series are transformed in log levels.
8 See Appendix C
9 The estimation procedure has been implemented with Matlab programming codes.
From Eq. (5), it follows that

\[ a_{21} + a_{23} - \gamma = 0 \]
\[ a_{22} + a_{24} + \gamma = 1 \]

These linear equality constraints for constrained likelihood objective function maximization have been imposed to obtain two values of \( \beta \) that differ for the sign. Considering Eq. (3), the positive value for price overshooting has been chosen. In our baseline model, the coefficients associated with the percentage of momentum traders and fundamentalists are fixed in time. However, this assumption can be relaxed. In fact, it is possible to construct a time-varying linear state-space model. We leave the integration of time varying parameters to future work.

5 Estimation results

Tables 1 and 2 report the maximum likelihood estimates of \( a_{22}, a_{24}, \gamma \) and \( \beta \) for equity prices and housing prices in the UK, France, Germany and the USA. The estimate of the model’s parameters with the cyclical conditions and the log-likelihood with the sample size are given in the four columns headed by the country name.

In all the countries considered, for equity prices (Table 1) the signs of \( a_{22} \) and \( a_{24} \) respect conditions in Eq. (11) for oscillatory phenomena. In particular, we have damped fluctuations \((-1 < a_{24} < 0)\) with \( a_{22} \) inside the allowed range size \((a_{22}^2 < -4a_{24})\). Moreover, all the estimated coefficients are statistically significant at 1% statistical level.

Looking at the percentage of the two different types of agent in the financial market, for France and Germany, we notice that the fundamentalists \((\gamma)\) are the minority in comparison with the momentum agents \((1 - \gamma)\). The opposite for the UK and the USA. Nevertheless, the percentage of the extrapolators is sufficiently high to have a significant impact on observed prices. In the UK, 75% of the agents are estimated to be fundamentalists while the 25% are extrapolators. In France and Germany, the momentum traders correspond to 71% and 54% respectively while the fundamentalists are estimated to be 29% and 46%. In the USA, 69% of agents are estimated to be fundamentalists and 31% extrapolators. It is worth noting that the percentage of fundamentalists and momentum agents is statistically significant at the 1% level for all the countries considered.

Once we obtain these results, from \( a_{22}, a_{24} \) and \( 1 - \gamma \), it is possible to obtain the value of \( \beta \) to analyze the price overshooting of the momentum agents. In the UK and the USA, even if the percentage of momentum traders is inferior in comparison with Germany and France, the price overshooting is higher. The highest price overshooting is in the UK (\( \beta = 3 \)), followed by the USA (\( \beta = 2.1 \)), Germany (\( \beta = 0.8 \)) and France (\( \beta = 0.4 \)). For example, in the UK, when the asset price is above (below) its value at the time before, extrapolative behavior implies that the economic agent optimistically (pessimistically) believes in a further price increase (decrease) of 3 times.

From the obtained results we notice similarities in the equity market across countries. Overall, in all the countries considered, the obtained results provide empirical support for Minsky’s hypothesis of the existence of financial cycles in equity prices as a consequence of the different price strategies defined in our model. However, in the UK and the USA
the percentage of extrapolators is lower compared to the fundamentalists, even if the price overshooting is higher in these two countries in comparison with Germany and France.

Table 1: Estimation via Kalman filter for equity prices

<table>
<thead>
<tr>
<th>Countries</th>
<th>France</th>
<th>Germany</th>
<th>UK</th>
<th>USA</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{22}$</td>
<td>1.0263***</td>
<td>0.9765***</td>
<td>1.0276***</td>
<td>0.9844***</td>
</tr>
<tr>
<td></td>
<td>(0.0005)</td>
<td>(0.0007)</td>
<td>(0.0007)</td>
<td>(0.0009)</td>
</tr>
<tr>
<td>$a_{24}$</td>
<td>-0.3146***</td>
<td>-0.4308***</td>
<td>-0.7707***</td>
<td>-0.6663***</td>
</tr>
<tr>
<td></td>
<td>(0.0029)</td>
<td>(0.0026)</td>
<td>(0.0047)</td>
<td>(0.0011)</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.29***</td>
<td>0.46***</td>
<td>0.75***</td>
<td>0.69***</td>
</tr>
<tr>
<td></td>
<td>(0.0027)</td>
<td>(0.0024)</td>
<td>(0.0039)</td>
<td>(0.0013)</td>
</tr>
<tr>
<td>$1 - \gamma$</td>
<td>0.71***</td>
<td>0.54***</td>
<td>0.25***</td>
<td>0.31***</td>
</tr>
<tr>
<td></td>
<td>(0.0027)</td>
<td>(0.0024)</td>
<td>(0.0039)</td>
<td>(0.0013)</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.4</td>
<td>0.8</td>
<td>3</td>
<td>2.1</td>
</tr>
</tbody>
</table>

| $\sigma_\varepsilon$ | 0.1702*** | 0.1302*** | 0.1401*** | 0.1482*** |
|                       | (0.0397)  | (0.0332)  | (0.0187)  | (0.0140)  |
| $\sigma_\eta$        | 0.2158*** | 0.2413*** | 0.1428*** | 0.1258*** |
|                       | (0.0197)  | (0.0230)  | (0.0273)  | (0.0289)  |

Cyclical Conditions

<table>
<thead>
<tr>
<th>条件</th>
<th>France</th>
<th>Germany</th>
<th>UK</th>
<th>USA</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[-1 &lt; a_{24} &lt; 0]$</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>$[a_{222} &lt; -4a_{24}]$</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
</tr>
</tbody>
</table>

Log-likelihood | 12.4351 | 16.3028 | 27.3701 | 28.6854 |
Sample size | 48 | 48 | 48 | 48 |

Notes: Standard errors in parentheses.

*, **, *** denotes statistical significance at the 10%, 5%, and 1% levels respectively.
Results of house prices are summarized in Table 2. For all the countries we find that both the sizes and the signs of $a_{22}$ and $a_{24}$ respect conditions number (11) for fluctuations. We have damped fluctuations for all the four countries considered ($a_{24} < 1$), with a value for France and USA near to one, likely to generate almost constant amplitude cycles. For the UK, France and the USA, both $a_{22}$ and $a_{24}$ are statistically significant at the 1% level. For Germany, the conditions for cycles are satisfied, but the coefficient estimates are not statistically significant.

For the UK, Germany and the USA the estimated share of fundamentalists ($\gamma$) is substantially higher than that of momentum traders. For the UK, 69% of agents are fundamentalists and the remaining 31% are extrapolators. In Germany, the momentum agents account for 30% while the fundamentalists are estimated to be 70%. In the USA, 74% of agents are estimated to be fundamentalists and the remaining 26% are extrapolators. Only for France do we find similar proportion for the momentum traders (51%) and fundamentalists (49%).

Again we can calculate the extent of price overshooting in the extrapolative pricing strategy. We find the highest price overshooting in the USA with a value of $\beta$ equal to 3.7. This value is followed by the price overshooting in the UK with France ($\beta = 1.9$) and in Germany ($\beta = 0.2$).

Overall, we find evidence for Minsky cycles on housing markets for the UK, France and the USA. For Germany, the point estimates for parameter suggest the presence of cyclical dynamics, however the relevant parameter is not statistically significant. Qualitatively speaking, these differences seem to be confirmed in the observed price’s series of the four countries: unlike the UK, France and the USA, the house price fluctuation in Germany is less evident (See Appendix D).

Comparing these results from the house market to those for the equity market, we find similarities between the two markets. With the exception of Germany, we find robust empirical evidence for Minsky’s hypothesis of the existence of financial cycles in a context of different price strategies in the two asset prices. In general, we notice a lower percentage of extrapolative agents compared with the fundamentalists with the highest price overshooting in the UK and the USA, the two advanced financial asset market-oriented economies. In this sense, the speculative position is primarily taken from beliefs that are not shared by the majority of the market.

Moreover, the obtained results confirm the importance of considering the housing prices affected by the presence of speculative forces that can generate cyclical fluctuations. The same forces of behavioral strategy that drive international financial markets also have the potential to affect other markets, like the housing market. In fact, it does not appear possible to explain the boom and bust in terms of fundamentals such as construction costs (Shiller, 2005; Shiller, 2007). The qualitative differences between equity asset and housing price can be detected by the smoothed estimate of the state variables, always obtained via the Kalman filter (See Appendix E).
Table 2: Estimation via Kalman filter for housing prices

<table>
<thead>
<tr>
<th>Countries</th>
<th>France</th>
<th>Germany</th>
<th>UK</th>
<th>USA</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{22}$</td>
<td>1.5102***</td>
<td>0.3580*</td>
<td>0.8991***</td>
<td>1.2195***</td>
</tr>
<tr>
<td></td>
<td>(0.0894)</td>
<td>(0.1935)</td>
<td>(0.0036)</td>
<td>(0.0220)</td>
</tr>
<tr>
<td>$a_{24}$</td>
<td>-0.9968***</td>
<td>-0.0583</td>
<td>-0.5924***</td>
<td>-0.9599***</td>
</tr>
<tr>
<td></td>
<td>(0.0090)</td>
<td>(0.1928)</td>
<td>(0.0040)</td>
<td>(0.0083)</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.49***</td>
<td>0.70***</td>
<td>0.69***</td>
<td>0.74***</td>
</tr>
<tr>
<td></td>
<td>(0.0885)</td>
<td>(0.0016)</td>
<td>(0.0032)</td>
<td>(0.0252)</td>
</tr>
<tr>
<td>$1 - \gamma$</td>
<td>0.51***</td>
<td>0.30***</td>
<td>0.31***</td>
<td>0.26***</td>
</tr>
<tr>
<td></td>
<td>(0.0885)</td>
<td>(0.0016)</td>
<td>(0.0032)</td>
<td>(0.0252)</td>
</tr>
<tr>
<td>$\beta$</td>
<td>1.9</td>
<td>0.2</td>
<td>1.9</td>
<td>3.7</td>
</tr>
<tr>
<td>$\sigma_{e}$</td>
<td>0.0621***</td>
<td>0.0305***</td>
<td>0.0830***</td>
<td>0.0374***</td>
</tr>
<tr>
<td></td>
<td>(0.0018)</td>
<td>(0.0025)</td>
<td>(0.0074)</td>
<td>(0.0032)</td>
</tr>
<tr>
<td>$\sigma_{\eta}$</td>
<td>0.0376***</td>
<td>0.0000</td>
<td>0.0670***</td>
<td>0.0673***</td>
</tr>
<tr>
<td></td>
<td>(0.0037)</td>
<td>(0.0000)</td>
<td>(0.0172)</td>
<td>(0.0061)</td>
</tr>
</tbody>
</table>

**Cyclical Conditions**

<table>
<thead>
<tr>
<th>Countries</th>
<th>France</th>
<th>Germany</th>
<th>UK</th>
<th>USA</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[-1 &lt; a_{24} &lt; 0]$</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>$[a_{222} &lt; -4a_{24}]$</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>Log-likelihood</td>
<td>67.8361</td>
<td>111.4751</td>
<td>56.297</td>
<td>77.1056</td>
</tr>
<tr>
<td>Sample size</td>
<td>48</td>
<td>48</td>
<td>48</td>
<td>48</td>
</tr>
</tbody>
</table>

Notes: Standard errors in parentheses.

*, **, *** denotes statistical significance at the 10%, 5%, and 1% levels respectively.
5 Conclusions

This paper has proposed a test of the asset price cycles based on the interaction of fundamentalist (mean reverting) and momentum valuation strategies. Both strategies are unobservable. The proposed model is formulated in a state space form and the parameters are estimated using the Kalman filter. We find robust empirical evidence for the presence of financial cycles in asset prices. Specifically, we find statistically significant evidence of financial cycles in the equity market for the UK, France, Germany and the USA. For the housing market we find strong evidence for the UK, France and the USA. We also find extrapolative price expectations overshoot more in market-based financial systems, namely the UK and the USA.

The results have both theoretical and empirical implications, contributing to the literature in two main aspects. Firstly, for debates in the Minskyan literature, our results support speculative Minskyan cycles in equity and real estate prices. This goes beyond the existing empirical Minsky literature which has so far only investigated debt cycles, but not asset prices cycles.

Secondly, our results support the behavioral theory, where heuristic decisions of agents are considered as the first source of instability and fluctuations in the economy (De Grauwe, 2012; Franke and Westerhoff, 2017). In this regard, the contribution of the present paper is to estimate the effect of this endogenous mechanism within the proposed analytical framework. Our results highlight a relevant aspect of the different price strategies, suggesting the fundamental role of extrapolative strategies in generating fluctuations both in the equity market (Beja and Goldman, 1980) and in the housing market (Dieci and Westerhoff, 2012; Bofinger et al., 2013). In other words, our results contrast with the standard theoretical approach to asset price fluctuations, based on rational expectations and market "fundamentals". Conversely, our findings are in line with the idea that price changes are not explained by an economic fundamental variation, but by the use of heuristics (Shiller, 2003).

Future research could aim to integrate other mechanisms in the framework proposed so as to improve the approximation of the asset price dynamics. Extensions of the baseline model proposed can be considered in a state space formulation with the Kalman filter approach. The model can be modified with time-varying coefficients of the measurement matrix. At the same time, other price strategies can be introduced in the model, like the adaptive price strategy among others. In this sense, an external exogenous variable representing the fundamental variable can be taken into account. For example, the profit for the equity market or the household income for housing prices. Moreover, in the model proposed, financial cycles are not linked to the real sector of the economy so future studies could be direct to the analysis of the relationship between the real and financial sector in a multivariate state space model setting. However, these modifications require an extension of the model proposed. These extensions should be explored in future research.

Finally, even if this task is beyond the scope of our paper, policy implications could be found if the momentum traders affect the rest of society by causing a cost of their actions. It is necessary to understand how to stabilize or control the financial fluctuations to avoid negative repercussions on the rest of the economy. In conclusion, we have to consider the role of the financial cycle and possible instability moving away from the idea that price changes always reflect rational and precise information in a permanent efficient financial market. Evidence from Minsky’s theory can help us to go in this direction.
Appendix A

In our model, the fundamentalists believe the asset price is determined solely by economic fundamentals which is known by them and updated with time. The fundamental value is intrinsic to the asset and based on the expected income stream. Following the efficient market hypothesis, the fundamentalist strategy can be defined in the following way

\[ P_t^f = P_{t-1}^f + \varepsilon_t \quad \varepsilon_t \sim N(0, \sigma_\varepsilon^2) \]  \hspace{1cm} (A.1)

Eq. (A.1) depicts a random walk. In other words, we assume that the fundamentalists believe that the price follows a random walk pattern. However, Eq. (A.1) is exactly a particular case of the more general mean reversion process

\[ P_t^f = P_{t-1} + \varphi(P_{t-1}^f - P_{t-1}) + \varepsilon_t \quad \varphi = 1, \quad \varepsilon_t \sim N(0, \sigma_\varepsilon^2) \]  \hspace{1cm} (A.2)

where \( \varphi \) measures the fundamentalists perceived speed of mean reversion of the market price towards the fundamental price. One implication of this is that, in the case of asset price boom or bust, fundamentalists expect market prices to revert to the fundamental value. Solving with \( \varphi = 1 \) we obtain

\[ P_t^f = P_{t-1} + \varphi P_{t-1}^f - \varphi P_{t-1} + \varepsilon_t \]

\[ P_t^f = P_{t-1}^f + \varepsilon_t \]

So, when \( \varphi = 1 \), we come back to our case.

Appendix B

Let us consider a discrete system

\[ U = [u_i(t)] = \begin{bmatrix} u_1(t) \\ \vdots \\ u_i(t) \\ \vdots \\ u_r(t) \end{bmatrix} \in \mathbb{R}^{rx1} \]

where

\[ u_i(t): \mathbb{R} \to \mathbb{R} \quad i = 1, \ldots, r \quad t \in [0, T] \]

We assume that:

Hp.1) functions \( u_i(t) \) can be described by their values assumed in discrete time. Introducing the vector

\[ U_j = [u_i(t_j)] \quad t_j = j\Delta t \quad j = 1, 2, \ldots, n \quad n\Delta t = T \]

Hp.2) the values at time \( t_j \) can be expressed by the values assumed at previous times \( t_{j-1}, \ldots, t_{j-R} \) where \( R \) is the memory’s degree. Introducing the vector
$[U_j] = \begin{bmatrix} U_j \\ U_{j-1} \\ U_{j-(R-1)} \end{bmatrix}$

the condition assumed by the second hypothesis can be expressed by

$[U_j] = \begin{bmatrix} U_j \\ U_{j-1} \\ U_{j-(R-1)} \end{bmatrix} = [A] \begin{bmatrix} U_{j-1} \\ U_{j-2} \\ U_{j-R} \end{bmatrix} = [A][U_{j-1}] \quad j = R + 1, \ldots, N$

where

$[U_k] \in \mathbb{R}^N \quad [A] \in \mathbb{R}^{N \times N} \quad N = rR$

It should be noted that it is necessary to know the state vector at the first $R$-times to activate the recursive law. Assuming in the previous equation $j = 1, \ldots, N$ (that amounts to assume that the state vector is known at $R$ previous times), the previous recursive law can be expressed by

$[U_2] = A[U_1]$
$[U_3] = A^2[U_1]$
$\ldots$
$[U_j] = A^{j-1}[U_1]$

Let be $V$ and $D$ the matrix of the eigenvectors and eigenvalues of the matrix $A$

$A = V D V^{-1} = I$

so that

$[U_j] = V D^{j-1} V^{-1} [U_1]$ 

also, the behavior of the recursive law is entirely described by the values of the eigenvalues

$\lambda_i \quad i = 1, \ldots, N$

When $\lambda_i \in \mathbb{R}, i = 1, \ldots, N$, the system is constant if $\lambda_i = 1 \forall i$, monotonic increasing (explosive oscillations) if $\lambda_i > 1$ for one $i$, monotonic decreasing (damped oscillations) if $\lambda_i < 1$ for one $i$.

In order to have an oscillating behavior it is necessary that

$\lambda_i \in \mathbb{C} \quad i = 1, \ldots, N$

Moreover, the behavior is depending on the modulus $\rho$ of the complex eigenvalues. Amplitude will be increasing, constant or decreasing if, respectively, $\rho$ is greater than equal or smaller than unity.

Now let us consider $r = 1$ and $R = 2$

$u_j = \alpha u_{j-1} + \beta u_{j-2}$

so that

$\begin{bmatrix} u_j \\ u_{j-1} \end{bmatrix} = A \begin{bmatrix} u_{j-1} \\ u_{j-2} \end{bmatrix}$

with
\[ A = \begin{bmatrix} \alpha & \beta \\ 1 & 0 \end{bmatrix} \]

We consider

\[ \det \begin{bmatrix} \alpha - \lambda & \beta \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - \alpha \lambda - \beta = 0 \]

so that the eigenvalues are

\[ \lambda_{1,2} = \frac{\alpha \pm \sqrt{\alpha^2 + 4\beta}}{2} \]

In order to have an oscillating behavior, the eigenvalues have to be complex so that

\[ \Delta = \alpha^2 + 4\beta < 0 \]

When this is the case:

\[ \lambda_{1,2} = \frac{\alpha}{2} \pm i \frac{\sqrt{-(\alpha^2 + 4\beta)}}{2} = a + ib \]

where \( i \) is the imaginary unit and \( a \) and \( b \) are real numbers. \( a \) is called the real part of the complex number and \( ib \) is the imaginary part. The complex number in the Cartesian form \( a + ib \) can be written in the equivalent trigonometric \( \rho(\cos \omega \pm i \sin \omega) \). The positive number \( \rho = (a^2 + b^2)^{\frac{1}{2}} \) is called the modulus of the complex number (Gandolfo, 2009).

In order to have oscillations of constant amplitude we require

\[ \rho = 1 \]

i.e.:

\[ \sqrt{\left(\frac{\alpha}{2}\right)^2 + \frac{-(\alpha^2 + 4\beta)}{4}} = 1 \]

from which

\[ \beta = -1 \]

Inserting in Eq. (B.1)

\[ -4 < -\alpha^2 \]

\[ \alpha^2 - 4 < 0 \]

\[ -2 < \alpha < 2 \]

Then, the conditions to have oscillating behavior of constant amplitude are

and

\[ \beta = -1 \]

\[ -2 < \alpha < 2 \]

If the condition in Eq. (B.1) is respected, with \(-1 < \beta < 0\) (length of eigenvalues < 1) we have damped oscillations. With \( \beta < -1 \) (length of eigenvalues > 1) we have explosive oscillations.
Connecting to our model with \( r = 2 \) and \( R = 2 \), where \( u_1 = p_f \) and \( u_2 = p_m \), we have

\[
\begin{pmatrix}
    u_{1,j} \\
    u_{2,j} \\
    u_{1,j-1} \\
    u_{2,j-1}
\end{pmatrix} =
\begin{pmatrix}
    a_{11} & 0 & 0 & 0 \\
    a_{21} & a_{22} & a_{23} & a_{24} \\
    1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
    u_{1,j-1} \\
    u_{2,j-1} \\
    u_{1,j-2} \\
    u_{2,j-2}
\end{pmatrix}
\]

\[
\begin{pmatrix}
    a_{11} - \lambda & 0 & 0 & 0 \\
    a_{21} & a_{22} - \lambda & a_{23} & a_{24} \\
    1 & 0 & -\lambda & 0 \\
    0 & 1 & 0 & -\lambda
\end{pmatrix} =
\begin{pmatrix}
    (a_{11} - \lambda) & 0 & 0 & 0 \\
    0 & (a_{22} - \lambda) & a_{23} & a_{24} \\
    1 & 0 & -\lambda & 0 \\
    0 & 1 & 0 & -\lambda
\end{pmatrix} = -(a_{11} - \lambda)(\lambda) \begin{pmatrix}
    1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0
\end{pmatrix} = 0
\]

The first two eigenvalues are

\[
\lambda_4 = a_{11} \in \mathbb{R} \quad \lambda_3 = 0
\]

Regarding the other eigenvalues, it should be noted that the problem is equivalent to the precedent case so that the system is oscillating if

\[
|a_{11}| \leq 1 \quad \forall a_{21}, \forall a_{23} \quad a_{24} < -\frac{a_{22}^2}{4}
\]

**Appendix C**

The Kalman filter is a recursive dynamic procedure for calculating the optimal estimator of the unobserved state vector. It is considered the best among the linear filters and one important advantage of using the state-space approach via the Kalman Filter is that stationarity of variables is not required. One limitation is that the state equation must be a first-order stochastic difference equation. However, it is often possible to rewrite a complicated dynamic process as a vector process (See for example Enders, 2016). The goal is to minimize the mean square prediction error of the unobserved state vector conditional of the observation of \( P_t \).

The optimal forecasting rule has the form

\[
Z_{t|t} = Z_{t|t-1} + K_t (P_t - P_{t|t-1})
\]

where \( K_t \) is a weight that changes as new information becomes available, \( Z_{t|t} \) denotes the forecast of state variable once \( P_t \) is realized while \( Z_{t|t-1} \) and \( P_{t|t-1} \) denote respectively the forecast of variables \( Z_t \) and \( P_t \) before \( P_t \) is realized.

Now we can select the optimal value of \( K_t \) to minimize the mean square prediction error at time \( t \)

\[
\min_{K_t} E_t (Z_t - Z_{t|t})^2 = \min_{K_t} E_t [Z_t - (Z_{t|t-1} + K_t (P_t - P_{t|t-1}))]^2
\]

using the equation (8) for the observable asset price, we obtain

\[
\min_{K_t} E_t \left[ Z_t - \left( Z_{t|t-1} + K_t (HZ_t - HZ_{t|t-1}) \right) \right]^2
\]

\[
\min_{K_t} E_t \left[ (I - HK_t) (Z_t - Z_{t|t-1}) \right]^2
\]

\[
\min_{K_t} (I - HK_t)^2 E_t (Z_t - Z_{t|t-1})^2
\]
Optimizing with respect to $K_t$ we get

$$-2H(I - HK_t)E_t(Z_t - Z_{t|t-1})^2 = 0$$

Indicating with $\Gamma_{t|t-1} = E_t(Z_t - Z_{t|t-1})^2$, we get

$$-2H(I - HK_t)\Gamma_{t|t-1} = 0$$

Solving for $K_t$, we obtain

$$K_t = \frac{H\Gamma_{t|t-1}}{H\Gamma_{t|t-1}H'}$$

Regrouping the equations, we obtain that

$$Z_{t|t-1} = AZ_{t-1|t-1}$$  \hspace{1cm} (C.1)

$$\Gamma_{t|t-1} = A\Gamma_{t-1|t-1}A' + Q$$  \hspace{1cm} (C.2)

$$P_{t|t-1} = HP_{t-1|t-1}$$

Equations (C.1) and (C.2) are the so-called prediction equations in the Kalman filtering.

The other equations we need are the three updating equations which are

$$K_t = \Gamma_{t|t-1}H'(\psi_t)^{-1}$$  \hspace{1cm} (C.3)

with

$$\psi_t = H\Gamma_{t|t-1}H'$$

$$Z_{t|t} = Z_{t|t-1} + K_t(P_t - P_{t|t-1})$$  \hspace{1cm} (C.4)

$$\Gamma_{t|t} = (I - K_tH)\Gamma_{t|t-1}$$  \hspace{1cm} (C.5)

In this case, the inference about $Z_t$ is updated using the observed value of $P_t$.

We start with a specification information set with initial conditions $Z_{0|0}$ and $\Gamma_{0|0}$. Then we use the prediction equations (C.1) and (C.2) to obtain $Z_{1|0}$ and $\Gamma_{1|0}$. Once we observe $P_1$ we use the updating equations (C.3), (C.4), and (C.5) to obtain $Z_{1|1}$, $\Gamma_{1|1}$ and $P_{1|1}$. We next use this information to form $Z_{2|1}$ and $\Gamma_{2|1}$, then forecasts are updated and we continue to repeat this process until the end of the dataset.

Given the vector prediction errors $U_T = (P_t - P_{t|t-1})$ and the variance-covariance matrix $\psi_t$, we can form the log-likelihood to be maximized and to estimate our parameters.

$$\log l = -\frac{T}{2} \ln(2\pi) - \frac{1}{2} \sum_{t=1}^{T} \ln(|\psi_{t|t-1}|) - \frac{1}{2} \sum_{t=1}^{T} \mu_t^T(\psi_{t|t-1})^{-1}\mu_t$$
Appendix D

Figure D1: Real equity prices index (1970-2017).

Figure D2: Real housing prices index (1970-2017).
Appendix E

The smoothed estimate of the state variables has been obtained via the Kalman filter. Smoothed states are estimated states at period t, which are updated using all available information. The results relative to the equity asset are reported in Figures E.3, E.4, E.5 and E.6. The results relative to housing price are reported in Figures E.7, E.8, E.9 and E.10.

In the figures below we have the smoothed state variable of the fundamentalists (red), the smoothed state variable of the extrapolative traders (blue), the observed asset prices (black) and the union of the three-time series. On the x-axis for the smoothed states of equity prices, we have the time period from 1973 to 2017, because the first three years of the sample period correspond to the observations required to initialize the Kalman filter and for which the smoothed states assume a value equal to zero. For the housing prices, in France, Germany and the USA we have the time period from 1972 to 2017. In the UK we have the time period from 1973 to 2017.
E.1 Equity Asset

Figure E1: Smoothed state variables (UK)
Figure E2: Smoothed state variables (France)

Figure E3: Smoothed state variables (Germany)
Figure E4: Smoothed state variables (US)

E.2 Housing Price
Figure E5: Smoothed state variables (UK)

Figure E6: Smoothed state variables (France)
Figure E7: Smoothed state variables (Germany)

Figure E8: Smoothed state variables (US)
Bibliography


