The Cambridge Controversies in the Theory of Capital:  
Revisiting the Reswitching Puzzle

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Abstract
When two techniques of production are compared, reswitching is the possibility that one technique can be cheapest at a low interest rate, switch to being more expensive at a higher rate, then reswitch to being cheapest at yet higher rates. Some think the inconsistency undermines the foundations of neoclassical economics.

The time value of money (TVM) equation is at the core of the reswitching puzzle. The equation takes the form of an nth order polynomial having n roots (interest rates). In most economic and financial analyses, including reswitching analysis, it is normal to use only one root. The remaining (n-1) roots are mostly complex or negative and they are usually ignored.

The new approach in this paper employs all n possible solutions for the interest rate to produce a new equation for the cost of production in which reswitching does not occur.

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1 Introduction

The reswitching puzzle is a part of the Cambridge controversies in capital theory. The controversies surfaced at the turn of the last century, intensified into the ‘Cambridge Controversies’ during the three decades after WWII, then died down and have simmered ever since. Harcourt (1972) provides a comprehensive survey of the controversies. Cohen and Harcourt (2003) provide a recent review.

When two techniques of production are compared, reswitching is the possibility that one technique can be cheapest at a low interest rate, switch to being more expensive at a higher rate, then reswitch to being cheapest at yet higher rates. For some this inconsistency undermines the foundations of neoclassical economic theory.

Samuelson (1966) explains the importance of reswitching thus:

*The phenomenon of switching back at a very low interest rate to a set of techniques that had seemed viable only at a very high interest rate involves more than esoteric technicalities. It shows the simple tale told by Jevons, Bohm-Bawerk, Wicksell, and other neoclassical writers – alleging that, as the interest rate falls in consequence of abstention from present consumption in favor of future, technology must become in some sense more “roundabout”, more “mechanized”, and “more productive” – cannot be universally valid.*

Nearly forty years later, Cohen and Harcourt (2003) concur.

*Looking back over this intellectual history, Solow (1963, p.10) suggested that “when a theoretical question remains debatable after 80 years there is a presumption that the question is badly posed – or very deep indeed.” Solow defended the “badly posed” answer, but we believe that the questions at issue in the recurring capital controversies are “very deep indeed.”*

This note contains a new approach to the puzzle. The approach depends on the fact that the analysis of reswitching employs the time value of money (TVM) equation and the associated processes of discounting and compounding. The TVM equation is a key equation in economics and finance. It takes the form of an nth order polynomial
having n roots (interest rates). In most economic and financial analyses, including the reswitching debate, it is usual to employ only one root, namely, the root yielding a positive, real interest rate. The remaining (n-1) roots are mostly complex or negative, and they are usually ignored. In this article it is argued that the unorthodox roots (interest rates) can shed light on the reswitching puzzle.

Section 2 describes a simple, numerical example of reswitching. Section 3, introduces an interim result about polynomials. In section 4 the interim result is used to produce a new expression for the cost of production that employs explicitly all possible interest rates, not just the orthodox. The exposure of the unorthodox solutions leads to a new perspective on reswitching, namely, that there is a simple relationship between the cost of production and all conceivable interest rates. Reswitching is not exhibited. Section 5 contains an application of the new equation to the numerical example. The final section is the conclusion.

2 An example of reswitching

Cohen and Harcourt illustrate reswitching using the example from Samuelson (1966). The same example is used here because it is well known and simple.

Table 1 shows the labor inputs for two production techniques, A and B, over three past time periods to produce one unit of output today. The wage rate is set at $1. For each technique, today’s cost of production depends on the past labor inputs and the rate of interest. The past labor inputs are compounded to today’s cost and seen as a form of capital (stored labor).

Table 1

<table>
<thead>
<tr>
<th>Time period</th>
<th>Labor inputs for technique A</th>
<th>Labor inputs for technique B</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (last period)</td>
<td>$L_1 = 0$</td>
<td>$L_1 = 6$</td>
</tr>
<tr>
<td>2 (two periods ago)</td>
<td>$L_2 = 7$</td>
<td>$L_2 = 0$</td>
</tr>
<tr>
<td>3 (three periods ago)</td>
<td>$L_3 = 0$</td>
<td>$L_3 = 2$</td>
</tr>
</tbody>
</table>
The equation that determines today’s cost is (1).

\[ C = L_1(1 + r) + L_2(1 + r)^2 + L_3(1 + r)^3 \]  

(1)

As the rate of interest falls, from above 100% through the range 100%-50% to below 50%, technique B is first more expensive than technique A, then switches to being cheaper, and finally reswitches to being more expensive again. Figure 1 illustrates the situation.

Figure 1. Ratio of costs (technique B to technique A) as the rate of interest changes

Reswitching is a puzzling feature of the relationship between cost and interest rate in equation (1). It is argued here that the relationship between \( C \) and \( r \) is subtler than it appears, the subtlety arising from the form of the function. Equation (1) is a polynomial therefore for each value of \( C \) there are three values of \( r \), not one value. Each value is as mathematically valid as another. In order to explore the part played by the hitherto discarded values an interim result is needed. The result enables a transformation of the cost function to one in which all interest rates are visible.
3 An interim result

Any TVM polynomial can be rearranged into the form of equation (2), which is called here the ‘special form’. The special form is characterized by the value $-1$ at the beginning, the value $1$ discounted over $n$ periods at the rate $z$ at the far end, and a range of discounted values, $b_i$, between the two. The whole is equal to zero.

$$-1 + \sum_{i=1}^{n} \frac{b_i}{(1 + z)^i} + \frac{1}{(1 + z)^n} = 0$$  \hspace{1cm} (2)

Many different arrays of values can exist for the parameters $b_i$. For any particular array of parameters, there are $n$ possible solutions for $z$. A special relationship (3) exists between a given array of parameters and all possible solutions for $z$ associated with that array.

$$\left| \sum_{i=1}^{n} b_i \right| = \prod_{i=1}^{n} |z|$$ \hspace{1cm} (3)

Equation (3) shows that the absolute value of the sum of the parameters in the special form (2) is equal to the product of the absolute values of all possible interest rates. This result is proved in the appendix and employed in the sections that follow.

4 The new expression for the cost of production

The new expression for the cost of production is obtained by comparing cost at one interest rate with cost at another rate. The initial situation is described by equation (1), repeated below, and the new situation by equation (4).

$$C = L_1(1 + r) + L_2(1 + r)^2 + L_3(1 + r)^3$$ \hspace{1cm} (1)

$$C' = L_1(1 + R) + L_2(1 + R)^2 + L_3(1 + R)^3$$  \hspace{1cm} (4)
In moving from equation (1) to (4) the interest rate changes from \( r \) to \( R \) and the cost of production from \( C \) to \( C' \). There are two ways to show the change in the interest rate – the additive and the multiplicative. The additive way is \((1+r) = (1+R+a)\) where \( a \) is an increment to an interest rate. It follows that \((r-R) = a\). The multiplicative way is \((1+r) = (1+R)(1+m)\) where \( m \) is an interest rate. From these two relationships the following result is deduced: \( \frac{a}{1+R} = m \). The new interest rate, \( R \), is assumed to be a positive real number. It has not been specified, however, whether the change is positive or negative, therefore the last result is restated as \( \frac{|a|}{1+R} = |m| \).

At this point the multiple interest rates are introduced. Equation (1) is a third-order polynomial; therefore there are three possible solutions for \( r \), each of which leads to the same cost of production. This fact begs a question that is not usually asked. When the interest rate pulls away from \( r \) and moves to the new level, \( R \), from which rate does it depart?

Orthodox analysis usually considers only the orthodox interest rate. In this analysis all possible rates are considered explicitly. This means there are three values of the increment, \( a \), or three values of the mark-up, \( m \), available to effect the change from the three values of \( r \) to the new interest rate, \( R \). It follows that \( \frac{|a_i|}{1+R} = |m_i| \) for \( i = 1 \) to \( 3 \).

All possible values of this expression are multiplied together to give equation (5).

\[
\prod \frac{|a_i|}{1+R}^3 = \prod |m_i| \quad \text{or} \quad \prod |r_i - R| = \prod |m_i| \quad (5)
\]

Keep this last equation in mind.

Now return to equation (1) that determines the cost of production. The expression for the multiplicative increment, \((1+R)(1+m)\), is substituted into equation (1), and the result rearranged into the special form. Equation (6) is the outcome.
\[
-1 - \frac{\left( \frac{L_2}{L_3} \right)}{(1 + m)} - \frac{\left( \frac{L_1}{L_3} \right)}{(1 + m)^2} + \frac{\left( \frac{C}{L_3} \right)}{(1 + m)^3} - 1 = 0 \quad (6)
\]

The special relationship applied to (6) implies (7).

\[
\left| \frac{\left( \frac{L_2}{L_3} \right)}{(1 + R)} - \frac{\left( \frac{L_1}{L_3} \right)}{(1 + R)^2} + \frac{\left( \frac{C}{L_3} \right)}{(1 + R)^3} - 1 \right| = \prod |m_i| \quad (7)
\]

Equation (4) for the new cost of production, \( C^* \), can be rearranged into (8).

\[
\left| \frac{\left( \frac{L_2}{L_3} \right)}{(1 + R)} - \frac{\left( \frac{L_1}{L_3} \right)}{(1 + R)^2} + \frac{\left( \frac{C^*}{L_3} \right)}{(1 + R)^3} - 1 \right| = 0 \quad (8)
\]

Solve for the first two elements on the left-hand side of (8) and substitute the result into (7) to obtain

\[
\frac{\left( \frac{C}{L_3} \right)}{(1 + R)} - \frac{\left( \frac{C^*}{L_3} \right)}{(1 + R)^3} = \prod |m_i|. \quad \text{Recall equation (5) and substitute it into this last result; then perform cancelations and rearrangements. The final result is the new equation for the cost of production, equation (9).}
\]

\[
|\Delta C| = L_3 \prod |a_i| \quad \text{or} \quad |\Delta C| = L_3 \prod |r_i - R| \quad (9)
\]

Equation (9) shows that the change in the cost of production is composed of two elements. The first element is the initial labor input (\( L_2 \) for technique A and \( L_3 \) for technique B). This input is grossed up by the second element, which is the product of all possible increments (two for technique A and three for B). Each increment is of the new interest rate relative to the original interest rates. The orthodox increment, \(|r_i-R|\), is but
one of the increments composing the second element. It follows that the relationship between the orthodox interest rate and the cost of production does not happen in isolation. The unorthodox interest rates, formerly ignored, also play a part.

It is worth restating this observation in another form in order to emphasize the result. The orthodox increment can be expressed as $|r_1 - R| = |\Delta r|$ which means that equation (9) can be restated as $|\Delta C| = L_3 \prod_{i=2}^{3} |r_i - R||\Delta r|$, i.e., the change in costs on the left-hand side is expressed as a function of the change in the orthodox interest rate on the far right-hand side. It is the relationship between these two orthodox changes that is graphed implicitly in Figure 1.

There is a snag however. The element that stands between the change in cost and the change in the orthodox interest rate ought to be a fixed parameter. Actually, it consists of $L_3$, which is a parameter, and the product of the unorthodox increments, which is not a parameter. The unorthodox increments contain $R$, the new interest rate; therefore their product is a variable. The all-inclusive product on the right hand side of equation (9) is a cluster of changes happening simultaneously.

The most important feature of equation (9) is what it reveals about the nature of the newfound relationship between the change in the cost of production and all possible increments in rates. The relationship between a change in the cost of production and the orthodox increment alone might not be simple; the relationship between a change in the cost of production and the product of all possible increments simultaneously, however, is always simple. The nature of the new equation makes it so. Ignoring all interest rates but the orthodox can give rise to reswitching; incorporating all possible interest rates into the analysis precludes it.
5  Numerical analysis

In this section the numbers from technique B in Samuelson’s example are fed into the new equation in order to illustrate the result. First, the equation is demonstrated at a fixed point, when a value for the interest rate results in a value for the cost of production. Secondly, the equation is demonstrated over a range of rates and costs.

The starting position is provided by equation (1). The quantities of labor and the orthodox interest rate are fed into the equation to produce the initial value of \( C \). Given \( C \) and the quantities of labor, the initial values of all interest rates are calculated. The new cost of production is then determined via equation (9) by the distances of the new rate, \( R \), from all the initial rates. As \( R \) slides up and down the real number line the three distances between \( R \) and the \( r_i \) change; the product of the three distances affects the distance of \( C^* \) from \( C \).

The starting position is wherever we care to set it. One possible starting position, a salient value of the orthodox interest rate, is 0. Technique B is captured in the third order polynomial \( C = 6(1+r) + 2(1+r)^3 \). If the interest rate is 0 then cost is 8. It is then possible to calculate all values of \((1+r)\) that solve the equation \( 8 = 6(1+r) + 2(1+r)^3 \). The three roots, or values of \((1+r)\), are 1 (which is known already) and the complex, conjugate pair \(-0.5 + 1.9365i \) and \(-0.5 - 1.9365i\). The interest rates \(|r_i|\) implied by these three values are 0, 2.4495 and 2.4495.

When the interest rate is 0, the orthodox equation (1) implies that \( C = \sum L_i \), i.e., the labor inputs escape compounding. Equation (9) becomes (9a).

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1 All numbers are calculated to many decimal places using Matlab but in this paper they are shown rounded to four decimal places.

2 Any starting position is valid. Some make more sense than others. Another salient position is the value of \( r \) that makes the cost of production zero, i.e., we solve for all values of \((1+r)\) in the equation \( 0 = 6(1+r) + 2(1+r)^3 \).

3 Notice the distinction between a root, a value of \((1+r)\) that solves a financial polynomial, and the value of \(|r|\) that is implied by it. The former can be a complex number; the latter is a distance between two numbers in the plane and therefore an absolute value and a real number.
\[ |C^* - \sum L_i| = L_3 \prod |r_i - R| \quad \text{or} \quad |C^* - 8| = 2 \prod |r_i - R| \] (9a)

The initial values of the interest rate, the \( r_i \), are known already, therefore all that is needed to establish a new cost of production is a value for the new interest rate, \( R \).
Assume the value is 0.5 or 50\%, i.e., the value of the root \((1+R)\) is 1.5.

This means the absolute values in the expression \( \prod |r_i - R| = \prod (1 + r_i) - (1 + R) \) are \(|1 - 1.5| = 0.5, |(-0.5 + 1.9365i) - 1.5| = 2.7839\) and \(|(-0.5 - 1.9365i) - 1.5| = 2.7839\) .
These values are fed into equation (9a).

\[ |C^* - \sum L_i| = L_3 \prod |r_i - R| \] (9a)

\[ |C^* - 8| = 2 \times 0.5 \times 2.7839 \times 2.7839 \quad \text{Therefore} \quad C^* = 15.75. \]

The value for \( C^* \) that results is the same as that from the orthodox calculation.
The difference lies in the route taken. The orthodox route to \( C^* \) is to feed \( R \) into the cost equation. The new route is to feed the differences between \( R \) and all the original values of \( r \) into equation (9) to give the difference between \( C^* \) and \( C \).

Next, the new equation is illustrated over a range. Figure 1 graphs the ratio of the two costs (B over A) against a range of orthodox interest rates. Figure 2 echoes Figure 1.
In Figure 2 the ratio of the two costs of production (B over A) is graphed against a range of values for \( \prod |r_i - R| \). The range of the product depicted on the horizontal axis of Figure 2 fully encompasses the range of \( R \) from 0\% to 200\% depicted in Figure 1. There is a switch in the range, but no reswitch.
Figure 2. Ratio of costs (technique B to technique A) as $\prod |r_i - R|$ changes with $R$.

Figure 3 below is an Argand diagram that shows the locations of the three roots $(1 + r_i)$ that satisfy equation (1) for technique B when $C=8$. The three roots embody all relevant information about the starting position. As $(1+R)$ slides up and down the real number line the distances vary between $(1+R)$ and the three roots $(1 + r_i)$. The distances represent all three possible changes in the interest rate. The product of the three rate changes, when multiplied by the initial input of labor, gives the change in the cost of production. In figure 3, the particular value shown for the interest rate, $(1+R)$, is 1.5, the value assumed in the numerical example above.

Orthodox analysis relates the cost of production to a value of the orthodox interest rate at a moment in time. It then asks what value of the cost of production would result if the orthodox interest rate were at a different level. Applying equation (1) followed by equation (4) gives a correct result for the change in the cost of production. But then the reasoning goes awry. There is an underlying assumption that the change in the cost of production is linked uniquely with the change in the orthodox interest rate only, that the change in the causal variable is limited to what happens along the real number line.
Figure 3. The roots \((1+r_i)\) of the cost equation (1) are shown in the complex plane. The roots are derived from the labor inputs of technique A when the cost of production is 8. The distances between each root and the new value of the interest rate \((1+R) = 1.5\) are shown as dotted lines. The unit circle is drawn to provide scale.
The assumption is wrong. Restricting the analysis to the real number line means that orthodox analysis does not tell the whole story. When we say that the interest rate changes it is important to be clear about what is changing. Equation (9) shows that a change in the cost of production is the result of the new interest rate moving away from all possible previous rates, not just the orthodox. When we change the interest rate we don’t change one thing, we change many things.

The implicit assumption in orthodox analysis means that the change in the cost of production is graphed against the change in the orthodox interest rate only. The variable on the vertical axis is compared with but one element of the true causal variable on the horizontal axis. The result is reswitching.

6 Conclusion

In this paper the reswitching phenomenon is re-examined in the context of Samuelson’s simple example. The phenomenon disappears when an expression containing a cluster of shifts in interest rates is brought into play, rather than the single orthodox shift. A new equation for the change in cost is derived; the equation defines the cluster and supports its use. The equation is easily generalized; it applies to a polynomial of any order. It is difficult to escape the conclusion that the unorthodox solutions to the time value of money equation, previously discarded, cannot be ignored.

Prior work shows that the unorthodox roots (interest rates) play a part in several other financial arenas. Dorfman (1981) uses all possible rates to produce a growth formula. In the context of fixed income mathematics Osborne (2005) produces a version of duration that is accurate and does not require convexity. In the context of capital budgeting Osborne (forthcoming) shows that net present value per dollar invested is composed of the mark-ups of all internal rates of return over the cost of capital, thereby shedding light on the debates about multiple IRRs and inconsistent ranking.

Reswitching is, therefore, a puzzle in a list of similar puzzles that exist in economics and finance, each of which causes debate in its own field. The debates have a
common factor – the TVM equation. The TVM equation is probably the most common
equation in finance, yet it has proved difficult to answer one of the simplest questions one
can ask of it, namely, what is the effect on value of a change in the interest rate? A
numerical answer is trivial; it is a simple and accurate algebraic answer that has proved
evasive. It is ironic that a trip into the complex plane is necessary to provide such an
answer. Whatever its intrinsic importance, one value of the reswitching puzzle may be
that it is an anomaly in the Kuhnian sense (Kuhn, 1962); it is an indicator of a wider
malaise in economics and finance, of a misunderstanding about precisely what happens
when interest rates change.

Questions remain. What is the meaning of the unorthodox interest rates, singly or
together? What are the implications of the analysis for the capital controversies overall?
Are there other anomalies or puzzles of a similar type in other arenas of economics and
finance that remain to be solved? Can the analysis be made stochastic? Answers to these
questions are left for future research. In the meantime a provocative perspective is
provided on a famous debate.
Appendix

Proof of a relationship between the coefficients and roots of the special form of the TVM equation

In the text it is stated that any TVM equation can be rearranged into a special form, which is repeated below for convenience.

\[-1 + \sum_{i=1}^{n} \frac{b_i}{(1+z)^i} + \frac{1}{(1+z)^n} = 0\]  \(2\)

It is asserted that the parameters and the roots of the special form are linked in a particular way: the absolute value of the sum of the parameters is equal to the product of the absolute values of the interest rates.

\[|\sum b_i| = \prod |z_i|\]  \(3\)

A proof of the statement is as follows. First, in the special form, set \(Z = (1+z)\) and multiply throughout by \(Z^n\), then factorize the equation and take absolute values. The result is:

\[|Z^n - b_1Z^{n-1} - ... - b_{n-1}Z - b_n - 1| = |Z - Z_1||Z - Z_2|....|Z - Z_{n-1}||Z - Z_n|\]

Now set \(Z = 1\). The left-hand side of becomes the absolute sum of the parameters, while the right hand side becomes the product of the absolute values of \(z_i\), since \(Z_i = (1+z_i)\) therefore \(|z_i| = |1 - Z_i|\). The proof is now complete.
References


